Math 180 Spring 09 Name:

TEST #2 @ 150 points

Write neatly. Show all work. **Write all proofs on separate paper. Label each exercise.**

1. Prove the following: *If a function f is differentiable at a point a, then it is continuous at a.*

2. Prove the following formula:

$$\frac{d}{dx}(\sin x) = \cos x$$

3. Find the derivative of each function and simplify as much as possible.
   You may use logarithmic differentiation when you find it necessary.

a) $y = (x^2 + \cos x + 1)^3$

b) $s = \dfrac{\sqrt{t}}{1+\sqrt{t}}$

c) $f(x) = 2x\sqrt{\sin x}$

d) $y = \theta^{-2}\sin^2(\theta^3)$

e) $r = \ln(3x) + 10e^{-\frac{x}{2}}$

f) $y = (x+1)^x$

4. Find the slope of the curve at the given point.

$y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$.

5. Find the critical numbers for the following functions:

a) $f(x) = \sqrt[3]{x^2 - x}$

b) $f(x) = x - 2\sin x$ on $[0, 2\pi]$

6. Find the absolute minimum and maximum values for the following functions on the given interval:

a) $f(x) = \dfrac{1}{x} + \ln x$, $x \in [0.5, 4]$

b) $g(\theta) = \sin\theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{5\pi}{6}$

7) Let $f(x) = xe^{-x}$. Do the following:

a) Graph the following function. Show: end behavior, behavior near vertical asymptotes (if any), intercepts, first and second derivative. Show all work and organize the information in a table, as we did in class. Label all points used.

b) What are the maximum and minimum values of the function?

c) What are the inflection points?

d) On what interval(s) is the function increasing? Decreasing?

e) On what interval(s) is the function concave up? Down?

8) Find the following limits:

a) $\lim_{x\to\infty}\left(\ln 2x - \ln(x+1)\right)$

b) $\lim_{x\to 0}(1-2x)^{\frac{1}{x}}$

c) $\lim_{x\to 0^+}\sqrt{x}\ln x$

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9. A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

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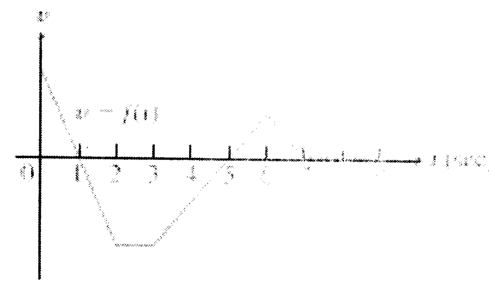
10. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

a) When does the particle move forward? Move backward?
b) When does the particle speed up? Speed down?
c) When is the particle's acceleration positive? Negative? Zero?
d) When does the particle move at its greatest speed?
e) When does the particle stand still for more than an instant?



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11 Suppose that the revenue is $r(x) = 9x$ and the cost of producing the items is $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

Extra Credit

1@3 points
Suppose the length of time, L, (in hours) that a drug stays in a person's system is a function of the dose or quantity administered, q, in mg. We have $L = f(q)$.

a) Interpret the statement $f(10) = 6$. Give units for 10 and for 6.

b) If $f'(10) = 0.5$, what are the units of 0.5 ?

c) Clearly interpret the statement $f'(10) = 0.5$ in terms of dose and duration.

#2@3points
Find a formula for the derivative of $(fgh)(x)$ in terms of the derivative of *f*, the derivative of *g*, and the derivative of *h*.

TEST 2 – SOLUTIONS

① Given: f – diff. at "a"

Prove f – cont. at "a"

Proof

We need to show that

$\lim_{x \to a} f(x) = f(a)$

We know f diff at "a" $\Rightarrow$

$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists

Let $f(x)-f(a) = \frac{f(x)-f(a)}{x-a} \cdot (x-a)$

true for any $x \neq a$

$\lim_{x \to a} (f(x)-f(a)) =$

$= \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a} (x-a)$

$= f'(a) \cdot 0$

$= 0$

So $\lim_{x \to a} (f(x)-f(a)) = 0 \Rightarrow$

$\lim_{x \to a} f(x) - f(a) = 0 \Rightarrow$

$\lim_{x \to a} f(x) = f(a) \Rightarrow$

f continuous at a

② $\frac{d}{dx}(\sin x) = \cos x$

Proof

Let $f(x) = \sin x$

Then $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$

$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h}$

$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\sin h \cos x}{h}$

$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$

$= \sin x \cdot 0 + \cos x \cdot 1$

$= \cos x$

Therefore,

$\frac{d}{dx}(\sin x) = \cos x$

(3) (a) $y=(x^2+\cos x+1)^3$

$y'=3(x^2+\cos x+1)^2(2x-\sin x)$

(b) $S=\dfrac{\sqrt{t}}{1+\sqrt{t}}$

Method I

$s'=\dfrac{\frac{1}{2\sqrt{t}}(1+\sqrt{t})-\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)}{(1+\sqrt{t})^2}$

$=\dfrac{\frac{1+\sqrt{t}}{2\sqrt{t}}-\frac{1}{2}}{(1+\sqrt{t})^2}=\dfrac{\frac{1+\cancel{\sqrt{t}}-\cancel{\sqrt{t}}}{2\sqrt{t}}}{(1+\sqrt{t})^2}$

$s'=\dfrac{1}{2\sqrt{t}(1+\sqrt{t})^2}$

Method II $S=\dfrac{\sqrt{t}+1-1}{1+\sqrt{t}}$

$S=1-\dfrac{1}{1+\sqrt{t}}$

$S'=-\dfrac{-\frac{1}{2\sqrt{t}}}{(1+\sqrt{t})^2}=\dfrac{1}{2\sqrt{t}(1+\sqrt{t})^2}$

(c) $f(x)=2x\sqrt{\sin x}$

$f'(x)=2\sqrt{\sin x}+\cancel{2}x\cdot\dfrac{1}{\cancel{2}\sqrt{\sin x}}\cdot\cos x$

$f'(x)=2\sqrt{\sin x}+\dfrac{x\cos x}{\sqrt{\sin x}}$

$f'(x)=\dfrac{2\sin x+x\cos x}{\sqrt{\sin x}}$

(d) $y=\theta^{-2}\sin^2(\theta^3)$

$y'=-2\theta^{-3}\sin^2(\theta^3)+$
$+\theta^{-2}\cdot 2\sin(\theta^3)\cdot\cos(\theta^3)\cdot 3\theta^2$

$y'=-2\theta^{-3}\sin^2(\theta^3)+6\sin(\theta^3)\cos(\theta^3)$

$y'=-2\theta^{-3}\sin^2(\theta^3)+3\sin(2\theta^3)$

(e) $r=\ln(3x)+10e^{-\frac{x}{2}}$

$r'=\dfrac{1}{3x}\cdot 3+10\cdot e^{-\frac{x}{2}}\left(\dfrac{-1}{2}\right)$

$r'=\dfrac{1}{x}-5e^{-\frac{x}{2}}$

(f) $y=(x+1)^x$ $\quad /\cdot\ln$

$\ln y=\ln(x+1)^x$

$\ln y=x\ln(x+1)$ $\quad /\cdot\frac{d}{dx}$

$\frac{1}{y}\cdot y'=\ln(x+1)+x\cdot\dfrac{1}{x+1}$

$y'=(x+1)^x\left(\ln(x+1)+\dfrac{x}{x+1}\right)$

(4) $x^2+y^2=y^4-2x \quad \Big/ \cdot \frac{d}{dx}$

$2x+2y\cdot y'=4y^3\cdot y'-2$

$2x+2=4y^3y'-2yy'$

$x+1=y'(2y^3-y)$

$y'=\dfrac{x+1}{2y^3-y}$

$m=y'\Big|_{\substack{x=-2\\ y=1}}$

$m=\dfrac{-2+1}{2(1)-1}=\dfrac{-1}{1}=-1$

$m=-1$

(5) (a) $f(x)=\sqrt[3]{x^2-x}=(x^2-x)^{\frac{1}{3}}$
$x\in\mathbb{R}$

$f'(x)=\frac{1}{3}(x^2-x)^{-\frac{2}{3}}(2x-1)$

$f'(x)=\dfrac{2x-1}{3\sqrt[3]{(x^2-x)^2}}$

$f'(x)=0$ when $2x-1=0$
$x=\frac{1}{2}$

$f'(x)$ undefined when
$x^2-x=0$
$x(x-1)=0 \;\Big\langle\; \begin{matrix} x=0 \\ x=1 \end{matrix}$

Critical numbers: $x=\frac{1}{2}, x=0, x=1$

(b) $f(x)=x-2\sin x$
$x\in[0,2\pi]$

$f'(x)=1-2\cos x$

$f'(x)=0$ when $1-2\cos x=0$
$\cos x=\frac{1}{2}$
$x=\frac{\pi}{3}$ OR $x=2\pi-\frac{\pi}{3}$

Critical numbers: $x=\frac{\pi}{3}, x=\frac{5\pi}{3}$

(6) (a) $f(x)=\frac{1}{x}+\ln x, \; x\in[0.5,4]$

1st. Find the critical numbers

$f'(x)=\dfrac{-1}{x^2}+\dfrac{1}{x}\,{}^{x/}$

$f'(x)=\dfrac{x-1}{x^2}$

$f'(x)=0$ when $x=1$ (CP)

$f'(x)$ undefined when $x=0$, but $0\notin[0.5,4]$

2nd Evaluate f at the C.P. and end points

$f(0.5)=\frac{1}{0.5}+\ln(0.5)=2-\ln 2\approx 1.3$

$f(1)=1+\ln 1=1$

$f(4)=\frac{1}{4}+\ln 4\approx 1.6$

Abs. min is 1 when $x=1$

Abs. max. is $\frac{1}{4}+\ln 4$ when $x=4$

(6) $g(\theta) = \sin\theta$, $\frac{-\pi}{2} \le \theta \le \frac{5\pi}{6}$

1st Find the critical numbers

$g'(\theta) = \cos\theta$

$\cos\theta = 0$ when

$\theta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

$k = -1$, $\theta = \frac{-\pi}{2}$ (C.P.)

$k = 0$, $\theta = \frac{\pi}{2}$

2nd evaluate y at C.P. and end points.

$g(-\frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$

$g(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$g(\frac{5\pi}{6}) = \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$

So Abs. min is -1 when $x = \frac{-\pi}{2}$

Abs. max. is 1 when $x = \frac{\pi}{2}$

(7) $f(x) = xe^{-x}$

x	$-\infty$	0	1	2	∞
f'	$+ + + +$		0	$- - - -$	
f	$-\infty \to$	$0 \to$	$\boxed{e^{-1}}$ max	$> 2e^{-2} >$	$y=0$ H.A
f''	$- - - -$			0	$+ +$

(f) Domain: $x \in \mathbb{R}$

x-in, y-in: $(0,0)$

$\lim_{x\to\infty} f(x) = \lim_{\infty} xe^{-x} = \infty \cdot 0!$

$= \lim_{\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$ (l'H)

$= \lim_{\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

so $y = 0$ H.A when $x \to \infty$

$\lim_{x\to-\infty} f(x) = \lim_{-\infty} xe^{-x}$

$= -\infty \cdot e^{\infty} = -\infty \cdot \infty$

$= -\infty$

(f') $f'(x) = e^{-x} + xe^{-x}(-1)$

$f'(x) = e^{-x}(1-x)$

$f'(x) = 0$ when $x = 1$ ($e^{-x} \neq 0$)

$f(1) = e^{-1}$

The sign of f' is given by the sign of $y = 1-x$ ($e^{-x} > 0$, any x)

$+ + \quad - -$

(f'') $f''(x) = e^{-x}(-1) - e^{-x}(1-x)$

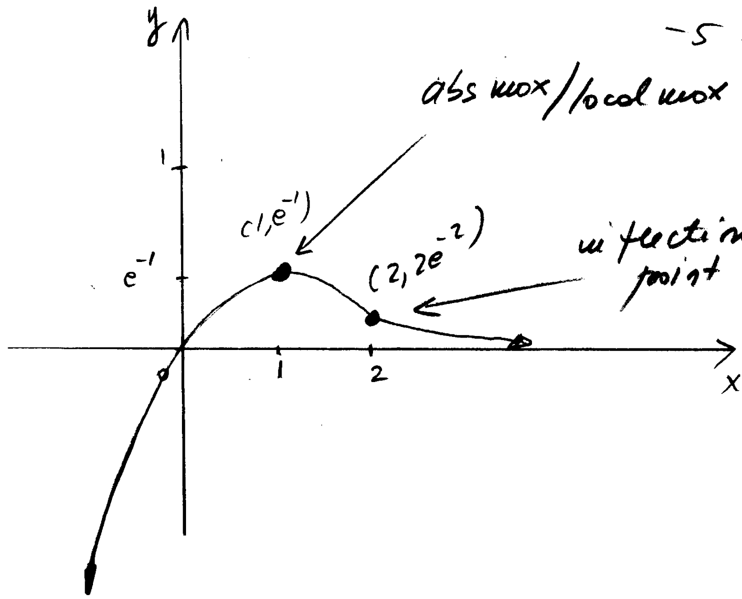
$f''(x) = e^{-x}(x-2)$

$f''(x) = 0$ when $x = 2$

$f(2) = 2e^{-2}$

The sign of f'' is given by the sign of $y = x-2$

$- \quad + \quad x$



$$\lim_{x\to 0} \frac{1}{x}\ln(1-2x) =$$

$$= \lim_{x\to 0} \frac{\ln(1-2x)}{x} = \frac{0}{0} \text{ (l'H)}$$

$$= \lim_{0} \frac{\frac{(-2)}{1-2x}}{1}$$

$$= \lim_{x\to 0} \frac{-2}{1-2x} = \frac{-2}{1} = -2$$

Therefore,

$$\lim_{x\to 0} (1-2x)^{\frac{1}{x}} = e^{-2}$$

(8)

(a) $\lim_{\infty} (\ln 2x - \ln(x+1)) =$

$$= \lim_{\infty} \ln \frac{2x}{x+1}$$

$$= \ln\left(\lim_{\infty} \frac{2x}{x+1}\right) \quad \frac{\infty}{\infty} \text{ (l'H)}$$

$$= \ln\left(\lim_{\infty} \frac{2}{1}\right) = \ln 2$$

(b) $\lim_{0} (1-2x)^{\frac{1}{x}} =$

$$= \lim_{0} e^{\ln(1-2x)^{\frac{1}{x}}}$$

$$= \lim_{0} e^{\frac{1}{x}\ln(1-2x)}$$

$$= e^{\lim_{0} \frac{1}{x}\ln(1-2x)}$$

Calculate $\lim_{0} \frac{1}{x}\ln(1-2x)$

(c) $\lim_{x\to 0^+} \sqrt{x}\ln x = 0\cdot(-\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \frac{\infty}{\infty} \text{ (l'Hopital)}$$

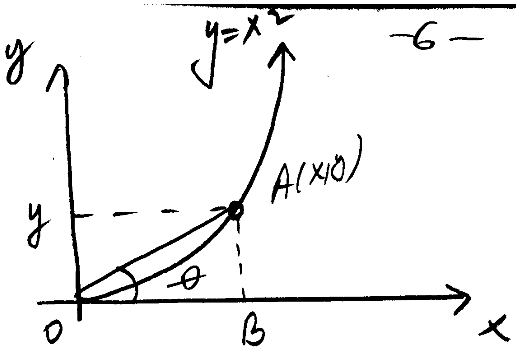
$$= \lim_{0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

$$= \lim_{0^+} -2x^{\frac{3}{2}-1}$$

$$= \lim_{0^+} -2x^{\frac{1}{2}} = -2(0) = 0$$

so $\lim_{0^+} \sqrt{x}\ln x = 0$

(9)



given: $\frac{dx}{dt} = 10$ m/sec

find $\frac{d\theta}{dt} = ?$

Solution

Method I

ΔOAB: $\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{x^2}{x}$

$\tan\theta = x \quad \Big/ \cdot \frac{d}{dt}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow$

$\frac{d\theta}{dt} = \cos^2\theta \cdot \frac{dx}{dt}$

when $x = 3$, $y = 9$

so $OB = 3$, $AB = 9$, and

$OA = \sqrt{9+81} = 3\sqrt{10}$

so $\cos\theta = \frac{OB}{OA} = \frac{3}{3\sqrt{10}} = \frac{1}{\sqrt{10}}$

Then $\frac{d\theta}{dt} = \frac{1}{10} \cdot 10$

$\frac{d\theta}{dt} = 1$ rad/sec

Method II

ΔOAB: $\tan\theta = x$

so $\theta = \tan^{-1} x \quad \Big/ \cdot \frac{d}{dt}$

$\frac{d\theta}{dt} = \frac{1}{1+x^2} \cdot \frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{1}{1+3^2} \cdot (10)$

$\frac{d\theta}{dt} = 1$ rad/sec

(10) (a) Particle moves forward when $v > 0$:

$t \in [0,1) \cup (5,7)$

Particle moves backward when $v < 0$:

$t \in (1,5)$

(b) Particle speeds up when v and a have the same sign:

$t \in (1,2) \cup (5,6)$

Particle speeds down when v and a have opposite signs:

$t \in [0,1) \cup (3,5) \cup (6,7)$

(c) $a = v'(t)$, so

$a > 0$ when $t \in (3,6)$

$a < 0$ when $t \in [0,2) \cup (6,7)$

$a=0$ when $t \in (2,3) \cup (7,9)$

(d) $t=0$ and $t \in [2,3]$

(e) $t \in [7,9]$

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(11) let $P(x)$ = profit
then $P(x) = r(x) - c(x)$

$P(x) = -x^3 + 6x^2 - 6x$

- Find the critical points:

$P'(x) = -3x^2 + 12x - 6$

$P'(x) = 0$ when $x = 2 \pm \sqrt{2}$

- Find if $x = 2-\sqrt{2}$ and $x = 2+\sqrt{2}$ are min. or max.

$P''(x) = -6x + 12$

$P''(2-\sqrt{2}) = 6\sqrt{2} > 0 \Rightarrow$
$x = 2-\sqrt{2}$ = min

$P''(2+\sqrt{2}) = -6\sqrt{2} < 0 \Rightarrow$
$x = 2+\sqrt{2}$ = max

So max. profit occurs at $x \approx 3.414$ thousand units and $P_{max} \approx 9.657$

EXTRA CREDIT

(1) (a) $f(q) = L$
$f(10) = 6$ so we have $q = 10$ and $L = 6$, so the units are 10 mg and 6 hrs.
The statement $f(10) = 6$ tells us that a dose of 10 mg will last for 6 hr.

(b) $L = f(q)$
$\frac{dL}{dq}$: the derivative
its units are the units of L over the units of q, or hr per mg; so 0.5 hr/mg

(c) $f'(10) = 0.5$ tells us that, at a dose of 10 mg, the rate of change of the duration is 0.5 hrs per mg; if we increase the dose by 1 mg, the drug would stay in the body about 30 min. longer

(2)

$$(fgh)(x) = (fg)(x)\,h(x) = [f(x)\,g(x)]\cdot h(x)$$

$$(fgh)'(x) = (f(x)\,g(x))'\,h(x) + [f(x)\,g(x)]\cdot h'(x)$$

$$= [f'(x)\,g(x) + f(x)\,g'(x)]\,h(x) + f(x)\,g(x)\,h'(x)$$

so

$$(fgh)'(x) = f'(x)\,g(x)\,h(x) + f(x)\,g'(x)\,h(x) + f(x)\,g(x)\,h'(x)$$