Review

4.1 Finding Critical Numbers. Finding Absolute Minimum and Maximum Values of a Function
4.4 Graphing a Function

Definition A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method

To find the absolute minimum and maximum values of a continuous function f on a closed interval [a,b]:

1. Find the critical numbers of f.
2. Find the values of f at the critical numbers and at the endpoints of the interval.
3. The largest of the values is the absolute maximum value; the smallest of the values is the absolute minimum value.

Exercise 1 Find the critical numbers of each function:

a) $f(x) = x^{\frac{3}{5}}(4 - x)$

b) $f(r) = \dfrac{r}{r^2 + 1}$

c) $f(z) = \dfrac{z + 1}{z^2 + z + 1}$

d) $f(x) = x^{\frac{4}{5}}(x - 4)^2$

e) $F(x) = \sqrt[3]{x^2 - x}$

f) $f(q) = \sin^2(2q)$

g) $g(q) = q + \sin q$

h) $f(x) = x \ln x$

Exercise 2 Find the absolute minimum and maximum values of each function on the given interval:

a) $f(x) = x - \sin x, \ \ x \in [0, 2p]$

b) $f(x) = \sqrt{9 - x^2}, \ \ x \in [-1, 2]$

c) $f(x) = x^2 + \dfrac{2}{x}, \ \ x \in \left[\dfrac{1}{2}, 1\right]$

d) $f(x) = \sin x + \cos x, \ \ x \in \left[0, \dfrac{p}{3}\right]$

e) $f(x) = x - 2\cos x, \ \ x \in [-p, p]$

f) $f(x) = x - 2\sin x, x \in [0, 3p]$

Exercise 3 Graph each function (as we did in class):

a) $f(x) = 2 - 2x - x^3$

b) $f(x) = \dfrac{x}{(1 + x)^2}$

c) $f(x) = \dfrac{\ln x}{\sqrt{x}}$

d) $f(x) = 2\cos x + \sin^2 x, \ \ x \in [-p, p]$

e) $f(x) = \dfrac{1 + x^2}{1 - x^2}$

f) $f(x) = x + \sqrt{1 - x}$

4.6 L'Hopital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Exercise 4

Find each limit. Use l'Hopital 's Rule where appropriate. (i)
If there is a more elementary method, consider it. (ii)
If l'Hopital's Rule doesn't apply, explain why. (iii)

a) $\lim_{x\to -1} \frac{x^2-1}{x+1}$

b) $\lim_{x\to 1} \frac{x^a-1}{x^b-1}$

c) $\lim_{x\to 0^+} \sqrt{x}\ln x$

d) $\lim_{x\to 0} \frac{\tan px}{\tan qx}$

e) $\lim_{x\to -\infty} x^2 e^x$

f) $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$

g) $\lim_{x\to 0^+} x^{\sin x}$

h) $\lim_{x\to 0} \frac{\cos mx - \cos nx}{x^2}$

i) $\lim_{x\to\infty} e^{-x}\ln x$

j) $\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$

k) $\lim_{x\to 0} \frac{1-e^{-2x}}{\sec x}$

l) $\lim_{x\to\infty} x^3 e^{-x^2}$

m) $\lim_{x\to\infty} \left(xe^{\frac{1}{x}} - x \right)$

n) $\lim_{x\to 0} (1-2x)^{\frac{1}{x}}$

o) $\lim_{x\to 1^+} (x-1)\tan\left(\frac{p x}{2}\right)$

p) $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

r) $\lim_{x\to 0^+} (-\ln x)^x$

s) $\lim_{x\to 0^+} x^2 \ln x$

Answers

Exercise 1: a) $0, 3/2$; b) ± 1; c) 0, -2; d) 0, 8/7, 4; e) 0, 1/2,1; f) $kp/4$, k integer; g) $(2k+1)p$, k integer; h) 1/e; i)

Exercise 2: a) abs. min: $f\left(\frac{p}{3}\right) = \frac{p}{3} - \sqrt{3}$, abs. max: $f\left(\frac{5p}{3}\right) = \frac{5p}{3} + \sqrt{3}$; b) abs. max: $f(0) = 3$, abs. min: $f(2) = \sqrt{5}$; c) abs. max: $f(2) = 5$, abs. min: $f(1) = 3$; d) abs. max: $f\left(\frac{p}{4}\right) = \sqrt{2}$, abs. min: $f(0) = 1$; e) abs. max: $f(p) = p + 2$, abs. min: $f\left(-\frac{p}{6}\right) = -\frac{p}{6} - \sqrt{3}$; f) abs. min: $f\left(\frac{p}{3}\right) = \frac{p}{3} - \sqrt{3} \approx -0.68$, abs. max: $f(3p) = 3p$

Exercise 4 a) ii -2; b) i a/b; c) i 0; d) i p/q; e) i 0; f)i 1; g) 1; h) i $\frac{n^2-m^2}{2}$; i) i 0; j) 2; k) iii 0; l) i 0; m) 1; n) e^{-2}; o) i $-2/p$; p) i 1/2 ; r) 1; s) i 0.