Math 160 Spring 2008 Name:

TEST #2 @ 175 points

Write neatly. Show all work. **Write all responses on separate paper.** **Clearly label the exercises.**

1. Prove that the statement is true for any natural number n.

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

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2. Evaluate the following:

a) $\sum_{k=1}^{10}\left(6-\frac{1}{2}k\right)$

b) $\sum_{k=1}^{25}\left(2^k-\frac{1}{2}\right)$

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3. Find the rational number whose decimal representation is $6.\overline{274}$.

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4. a) Give an example of an arithmetic sequence and find the sum of the first 50 terms.

b) Give an example of a geometric sequence and find the sum of the first 100 terms.

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5. Find the partial fraction decomposition:

a) $\frac{37-11x}{(x+1)(x^2-5x+6)}$

b) $\frac{x^2}{(x^2+4)^3}$

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6. a) Draw the vector $\vec{v}$ that goes from the origin to the point $(-3,4)$.

b) Write the vector $\vec{v}$ in component form $<a,b>$.

c) Write the vector $\vec{v}$ in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

d) Find the magnitude of the vector.

e) Find the angle θ, $0^\circ \le \theta < 360^\circ$ that the vector makes with the positive x-axis.

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7. Show that the two given vectors are parallel and determine whether they have the same direction or opposite directions.

$a=<6,18>, b=<-4,-12>$

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8. Solve the following equations.

a) $\sin^2 x - \sin x - 1 = 0$; solve in $[0, 2\pi)$ and approximate the solutions to four decimal places.

b) $\sin 2x = -1.5\cos x$; solve in $[0, 2\pi)$ and approximate the solutions to four decimal places.

c) $\cos x + 1 = 2\sin^2 x$; find all real solutions.

d) $\ln(\sin x) = 0$; find all real solutions.

e) $\cos\left(3x - \frac{\pi}{4}\right) = 0$; solve in $[0, 2\pi)$.

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9. a) Find a formula for $\sin\frac{x}{2}$ in terms of $\cos x$.

b) Find a formula for $\cos 3x$ in terms of $\cos x$.

c) Find a formula for $\tan(a+b)$ in terms of $\tan a$ and $\tan b$.

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10. Prove the following identities:

a) $\sqrt{\frac{1-\sin x}{1+\sin x}} = \frac{|\cos x|}{1+\sin x}$

b) $\sin 4a = 4\sin a\cos^3 a - 4\cos a\sin^3 a$

c) $\ln\sec t = -\ln\cos t$

EXTRA CREDIT

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11. Let **a**, ***b***, and ***c*** three vectors. Prove the following properties:

a) $a \cdot a = \|a\|^2$

b) $a \cdot (b + c) = a \cdot b + a \cdot c$

c) $a + b = b + a$

160

TEST # 2 - SOLUTIONS

Use Math. Induction:

(1) Let $P_n: 1\cdot2+2\cdot3+\dots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

Step 1 Check P_1 is true

$P_1: \quad 1\cdot2=\frac{1(1+1)(1+2)}{3}$

$2=\frac{1\cdot2\cdot3}{3}$

$2=2$ true

Step 2 Assume P_k true for some k

$1\cdot2+2\cdot3+\dots+k(k+1)=\frac{k(k+1)(k+2)}{3}$

Prove P_{k+1} true

$1\cdot2+2\cdot3+\dots+k(k+1)+(k+1)(k+2)=$

$=\frac{(k+1)(k+2)(k+3)}{3}$

$\underbrace{1\cdot2+2\cdot3+\dots+k(k+1)}_{\text{induction hypothesis}}+(k+1)(k+2)=$

$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)$

$=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$

$=\frac{(k+1)(k+2)(k+3)}{3}$

So P_{k+1} is true

Therefore,

$1\cdot2+2\cdot3+\dots+n(n+1)=\frac{n(n+1)(n+2)}{3}$

for any $n\geq1$

(2) (a) $\sum_{k=1}^{10}\left(6-\frac{1}{2}k\right)=$

$=\sum_{k=1}^{10}6-\sum_{k=1}^{10}\frac{1}{2}k$

$=\sum_{k=1}^{10}6-\frac{1}{2}\underbrace{\sum_{k=1}^{10}k}$

(arithmetic series
$a_1=1,\ a_{10}=10$
$S_{10}=\frac{(a_1+a_{10})10}{2}=\frac{11\cdot10}{2}$
$S_{10}=55$)

$=6\cdot10-\frac{1}{2}\cdot55$

$=60-\frac{55}{2}=\frac{120-55}{2}=\frac{65}{2}$

(b) $\sum_{k=1}^{25}\left(2^k-\frac{1}{2}\right)=\underbrace{\sum_{k=1}^{25}2^k}-\sum_{k=1}^{25}\frac{1}{2}$

(geometric series
$a_1=2$
$r=2$
$S_{25}=\frac{2(1-2^{25})}{1-2}$
$S_{25}=-2(1-2^{25})$)

$=-2(1-2^{25})-\frac{1}{2}\cdot25$

$=-2+2^{26}-\frac{25}{2}$

$=2^{26}-\frac{29}{2}$

(3) $6.\overline{274} = 6.274274\ldots$

$$= \underbrace{6 + \frac{274}{10^3} + \frac{274}{10^6} + \frac{274}{10^9} + \cdots}$$

S = infinite geometric series with $a_1 = \frac{274}{1000}$

$$r = \frac{1}{1000}$$

$|r| < 1$, so $S = \frac{a_1}{1-r}$.

$$S = \frac{\frac{274}{10^3}}{1 - \frac{1}{10^3}}$$

So, $6.\overline{274} = 6 + \dfrac{\frac{274}{1000}}{1 - \frac{1}{1000}}$

$$= 6 + \frac{274}{999} = \frac{6268}{999}$$

$$6.\overline{274} = \frac{6268}{999}$$

(4) (a) Start with $a_1 \in \mathbb{R}$ keep adding the same constant d over and over

$a_1, a_1+d, a_1+2d, \ldots$

$$a_n = a_1 + (n-1)d$$

$$S_n = \frac{(a_1 + a_n)\,n}{2}$$

(b) Start with $a_1 \in \mathbb{R}$, $a_1 \neq 0$ and keep multiplying by the same nonzero constant r over and over

$a_1, a_1 r, a_1 r^2, \ldots$

$$a_n = a_1 r^{n-1}$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

(5) (a) $$\frac{37-11x}{(x+1)(x^2-5x+6)} = \frac{37-11x}{(x+1)(x-3)(x-2)}$$

$$= \frac{\overset{(x-3)(x-2)}{A}}{x+1} + \frac{\overset{(x+1)(x-2)}{B}}{x-3} + \frac{\overset{(x+1)(x-3)}{C}}{x-2}$$

$$37-11x = A(x-3)(x-2) + B(x+1)(x-2) + C(x+1)(x-3)$$

if $x = 3$, $37-33 = B(4)(1) \Rightarrow B = 1$

if $x = -1$, $37+11 = A(-4)(-3) \Rightarrow A = +4$

if $x = 2$, $37-22 = C(3)(-1) \Rightarrow C = -5$

Therefore,

$$\frac{37-11x}{(x+1)(x^2-5x+6)} = \frac{4}{x+1} + \frac{1}{x-3} - \frac{5}{x-2}$$

(b) $$\frac{x^2}{(x^2+4)^3} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$$

$(x^2+4)^2$ $\quad -3 - \quad x^2+4$

$$x^2 = (Ax+B)(x^2+4)^2 + (Cx+D)(x^2+4) + Ex+F$$

$$x^2 = (Ax+B)(x^4+8x^2+16) + (Cx+D)(x^2+4) + Ex+F$$

$$\begin{aligned} x^2 = Ax^5 + {} & 8Ax^3 + 16Ax \\ & + Bx^4 + 8Bx^2 + 16B \\ & + Cx^3 + 4Cx \\ & + Dx^2 + 4D \\ & + Ex + F \end{aligned}$$

$A = 0$

$B = 0$

$8A + C = 0 \Rightarrow C = 0$

$8B + D = 1 \Rightarrow D = 1$

$16A + 4C + E = 0 \Rightarrow 4C + E = 0 \Rightarrow E = 0$

$16B + 4D + F = 0 \Rightarrow 4 + F = 0 \Rightarrow F = -4$

So, $$\frac{x^2}{(x^2+4)^3} = \frac{1}{(x^2+4)^2} - \frac{4}{(x^2+4)^3}$$

(6) (a)

(b) $\bar{v} = \langle -3, 4 \rangle$

(c) $\bar{v} = -3\bar{i} + 4\bar{j}$

(d) $\|\bar{v}\| = \sqrt{3^2+4^2} = 5$

(e) $\tan\theta = \frac{y}{x} = \frac{-4}{3}$

$\hat{\theta} = \tan^{-1}(\frac{-4}{3}) = -53°$

so θ, the angle that $\bar{v}$ makes with positive x-axis is $180° - 53° = 127°$

$\theta = 127°$.

(7) $a = \langle 6, 18 \rangle$

$b = \langle -4, -12 \rangle$

$\cos\theta = \frac{a \cdot b}{\|a\| \|b\|}$

$a \cdot b = -24 - 216 = -240$

$\|a\| = \sqrt{6^2+18^2} = \sqrt{360} = 6\sqrt{10}$

$\|b\| = \sqrt{4^2+12^2} = \sqrt{160} = 4\sqrt{10}$

$\cos\theta = \frac{-240}{6\sqrt{10} \cdot 4\sqrt{10}} = \frac{-24}{24} = -1$

$\cos\theta = -1 \Rightarrow \theta = 180°$,

so the vectors are parallel and have opposite directions.

(8) (a) $\sin^2 x - \sin x - 1 = 0$

quadratic equation in $\sin x$

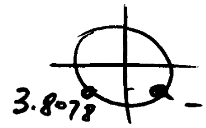
$\sin x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\sin x = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$

$\sin x = \frac{1-\sqrt{5}}{2}$ OR $\begin{cases} \sin x = \frac{1+\sqrt{5}}{2} > 1 \\ \text{not possible.} \end{cases}$

$x = \sin^{-1}\frac{1-\sqrt{5}}{2} = -0.6662$

$\notin [0, 2\pi)$



so $x_1 = 2\pi - 0.6662 = 5.6170$.

OR

$x_2 = \pi + 0.6662 = 3.8078$

$x \in \{5.6170, 3.8078\}$

(b) $\sin 2x = -1.5\cos x$

$2\sin x \cos x + 1.5\cos x = 0$

$\cos x(2\sin x + 1.5) = 0$

$\cos x = 0$ OR $2\sin x = -1.5$

$x = \frac{\pi}{2} \approx 1.5708$

OR

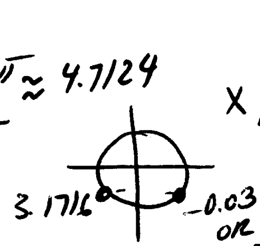
$x = \frac{3\pi}{2} \approx 4.7124$

$\sin x = \frac{-1.5}{2}$

$\sin x = -\frac{3}{4}$

$x_1 = \sin^{-1}(-\frac{3}{4}) = -0.8481$

$\notin [0, 2\pi)$



$x \in \begin{Bmatrix} 1.5708, 4.7124, \\ 5.4357, 3.9897 \end{Bmatrix}$

so $x = 2\pi - 0.8481 = 5.4351$

OR

$x = \pi + 0.8481 = 3.9897$

(c) $\tan(a+b) = \dfrac{\sin(a+b)}{\cos(a+b)}$

$$= \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b}$$

$$= \frac{\dfrac{\sin a \cancel{\cos b}}{\cos a \cancel{\cos b}} + \dfrac{\sin b \cancel{\cos a}}{\cancel{\cos a} \cos b}}{\dfrac{\cos a \cos b}{\cos a \cos b} - \dfrac{\sin a \sin b}{\cos a \cos b}}$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

(10) (a)

$$LHS = \sqrt{\frac{1-\sin x}{1+\sin x}} = \sqrt{\frac{(1-\sin x)(1+\sin x)}{(1+\sin x)^2}}$$

$$= \sqrt{\frac{1-\sin^2 x}{(1+\sin x)^2}} = \frac{\sqrt{\cos^2 x}}{\sqrt{(1+\sin x)^2}} = \frac{|\cos x|}{1+\sin x}$$

$= RHS$

Note that $1+\sin x > 0$

so $\sqrt{(1+\sin x)^2} = 1+\sin x$

Therefore, the given eq. is an identity

(b) $RHS = 4\sin a \cos^3 a - 4\cos a \sin^3 a$

$= 4\sin a \cos a(\cos^2 a - \sin^2 a)$

$= 2(2\sin a \cos a)(\cos 2a)$

$= 2(\sin 2a)(\cos 2a)$

$= \sin(4a)$

$= LHS$

So, the given equation is an identity

(c) $LHS = \ln(\sec t)$

$= \ln\left(\frac{1}{\cos t}\right)$

$= \ln 1 - \ln \cos t$

$= 0 - \ln \cos t$

$= -\ln \cos t = RHS$

So, the given equation is an identity

(11) (a) $a \cdot a = \|a\|^2$

Proof

Let $a = \langle a_1, a_2 \rangle$

$a \cdot a = a_1^2 + a_2^2 = \|a\|^2$

(b) $a \cdot (b+c) = a \cdot b + a \cdot c$

Proof

Let $a = \langle a_1, a_2 \rangle$
$b = \langle b_1, b_2 \rangle$
$c = \langle c_1, c_2 \rangle$

$a \cdot (b+c) = \langle a_1, a_2 \rangle \cdot \langle b_1+c_1, b_2+c_2 \rangle$
$= a_1(b_1+c_1) + a_2(b_2+c_2)$
$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2$
$= (a_1 b_1 + a_2 b_2) + (a_1 c_1 + a_2 c_2)$
$= a \cdot b + a \cdot c$

(c) $a+b = b+a$

$a+b = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle$
$= \langle a_1+b_1, a_2+b_2 \rangle$
$= \langle b_1+a_1, b_2+a_2 \rangle$
$= \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle$
$= b + a$

(c) $\cos x + 1 = 2\sin^2 x$

$\cos x + 1 = 2(1-\cos^2 x)$

$\cos x + 1 = 2 - 2\cos^2 x$

$2\cos^2 x + \cos x - 1 = 0$

$$\cos x = \frac{-1 \pm \sqrt{1+4(2)}}{2(2)}$$

$$\cos x = \frac{-1 \pm 3}{4} < \begin{matrix} -1 \\ \frac{1}{2} \end{matrix}$$

$\cos x = \frac{1}{2}$ OR $\cos x = -1$

$x = \frac{\pi}{3} + 2\pi k$ OR $x = \frac{5\pi}{3} + 2\pi k$

$x = \pi + 2\pi k$

$x \in \{\frac{\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k, \pi + 2\pi k \,/\, k \in \mathbb{Z}\}$

(d) $\ln(\sin x) = 0$

Condition: $\sin x > 0$

$e^0 = \sin x$

$\sin x = 1$

$x = \frac{\pi}{2} + 2\pi k, \ k \in \mathbb{Z}$

(e) $\cos(3x - \frac{\pi}{4}) = 0$

$3x - \frac{\pi}{4} = \frac{\pi}{2} + \pi k$

$3x = \frac{\pi}{2} + \frac{\pi}{4} + \pi k$

$3x = \frac{3\pi}{4} + \pi k$

$x = \frac{\pi}{4} + \frac{\pi k}{3}, \ k \in \mathbb{Z}$

if $k=0$, $x = \frac{\pi}{4}$

$k=1$, $x = \frac{\pi}{4} + \frac{\pi}{3} = \frac{7\pi}{12}$

$k=2$, $x = \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$

$k=3$, $x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$k=4$, $x = \frac{\pi}{4} + \frac{4\pi}{3} = \frac{19\pi}{12}$

$k=5$, $x = \frac{\pi}{4} + \frac{5\pi}{3} = \frac{23\pi}{12}$

$x \in \{\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}\}$

(9) (a) $\cos 2a = 1 - 2\sin^2 a$

so, $\cos x = 1 - 2\sin^2 \frac{x}{2}$

$2\sin^2 \frac{x}{2} = 1 - \cos x$

$\sin^2 \frac{x}{2} = \frac{1-\cos x}{2}$

$\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

(b) $\cos 3x = \cos(x + 2x)$

$= \cos x \cos 2x - \sin x \sin 2x$

$= \cos x(2\cos^2 x - 1) - \sin x(2\sin x \cos x)$

$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$

$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$

$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$