Math 150 Spring 08 Name:

TEST #2 @ 150 points

Solve the problems on separate paper. Clearly label the problems. Show all steps in order to get credit. No proof, no credit given

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1. Graph $f(x) = \cos x$ and $f^{-1}(x) = \cos^{-1}(x)$ on the same coordinate system, showing the relation between the two graphs (symmetry about the line $y = x$). Answer the following questions:

   a) What is the domain and range of $f(x) = \cos x$?

   b) What is the domain and range of $f^{-1}(x) = \cos^{-1}(x)$?

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2. a) Graph $y = 1 + 2\cos(3x)$ between 0 and 2π. Identify the amplitude and period and label the axes accurately.

   b) Find the x-intercepts of the graph within the period graphed; that is, solve the equation $y = 0$ in $[0, 2\pi]$. Give exact answers as well as approximations.

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3. Graph $y = \frac{1}{2}\sin\left(2x + \frac{2\pi}{3}\right)$ over one period. Identify the amplitude , period, and phase shift and label the axes accurately.

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4. Use addition of y-coordinates to sketch the graph of $y = x + \sin x$ between $x = 0$ and $x = 2\pi$.

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5. Evaluate the following. Give exact answers.

   a) $\sin^{-1}(-1)$

   b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

   c) $\tan^{-1}(-1)$

   d) $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)$

   e) $\cos^{-1}\left(\cos\frac{7\pi}{4}\right)$

   f) $\sin\left(\cos^{-1}\frac{1}{3}\right)$

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6. Prove the following identities

a) $\cot x - \tan x = \dfrac{\cos 2x}{\sin x \cos x}$

b) $\dfrac{1}{1-\sin t} + \dfrac{1}{1+\sin t} = 2\sec^2 t$

7. Find a formula for $\sin 3a$ in terms of $\sin a$.

8. Solve the following equations.
When appropriate, show EXACT answers.
ONLY when NO exact answer is possible, express solutions rounded to two decimal places.

a) Find ALL real solutions: $2\sin 2x + 1 = 0$

b) Solve on $[0, 2\pi)$: $\cos(4x) = 1$

c) Solve on $[0, 2\pi)$: $3\tan\theta - 3 = 0$

d) Solve on $[0, 2\pi)$: $\cos\theta = -0.4$

e) Find ALL real solutions: $2\sin^2 x - \sin x - 1 = 0$

f) Find ALL real solutions: $\cos x - \cos 2x = 0$

g) Solve on $[0, 2\pi)$: $4\sin^2 a + 4\cos a - 5 = 0$

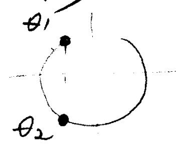
(d) $\cos\theta = -0.4$ in $[0, 2\pi)$

$\theta_1 = \cos^{-1}(-0.4) \approx 1.98$

OR

$\theta_2 = 2\pi - \theta_1 \approx 4.3$

$\theta \in \{1.98, 4.3\}$



(e) $2\sin^2 x - \sin x - 1 = 0$

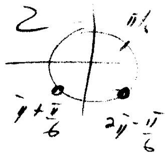
$\sin x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\sin x = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$

$\sin x = \frac{1 \pm 3}{4} \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

$\sin x = 1$ OR $\sin x = -\frac{1}{2}$

$x = \frac{\pi}{2} + 2\pi k$



$x = \frac{7\pi}{6} + 2\pi k$

OR

$x = \frac{11\pi}{6} + 2\pi k$

$$\begin{cases} x = \frac{\pi}{2} + 2\pi k \\ \text{OR} \\ x = \frac{7\pi}{6} + 2\pi k \\ \text{OR} \\ x = \frac{11\pi}{6} + 2\pi k \end{cases} \quad k \in \mathbb{Z}$$

(f) $\cos x - \cos 2x = 0$

$\cos x - (2\cos^2 x - 1) = 0$

$\cos x - 2\cos^2 x + 1 = 0 \quad /(-1)$

$2\cos^2 x - \cos x - 1 = 0$

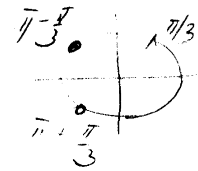
$\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\cos x = \frac{1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)}$

$\cos x = \frac{1 \pm 3}{4} \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

$\cos x = 1$ OR $\cos x = -\frac{1}{2}$

$x = 2\pi k$



$x = \frac{2\pi}{3} + 2\pi k$

OR

$x = \frac{4\pi}{3} + 2\pi k$

$$\begin{cases} x = 2\pi k \\ \text{OR} \\ x = \frac{2\pi}{3} + 2\pi k \\ \text{OR} \\ x = \frac{4\pi}{3} + 2\pi k \end{cases} \quad k \in \mathbb{Z}$$

(g) $4\sin^2 a + 4\cos a - 5 = 0$ in $[0, 2\pi)$

$4(1 - \cos^2 a) + 4\cos a - 5 = 0$

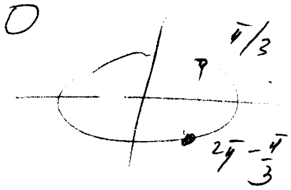
$-4\cos^2 a + 4\cos a - 1 = 0$

$4\cos^2 a - 4\cos a + 1 = 0$

$(2\cos a - 1)^2 = 0$

$2\cos a - 1 = 0$

$\cos a = \frac{1}{2}$



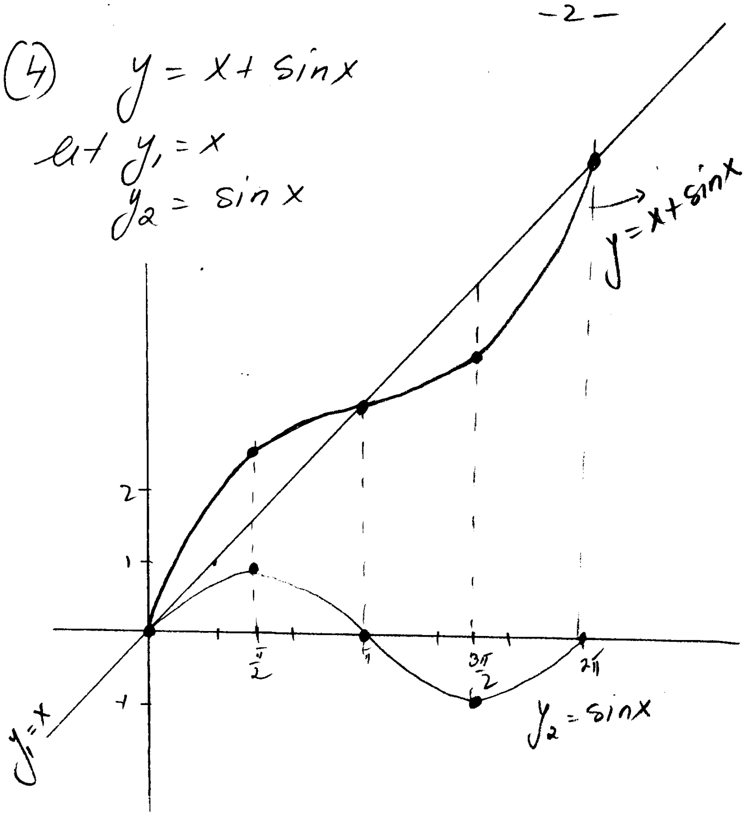
$a = \frac{\pi}{3}$ OR $\frac{5\pi}{3}$

$a \in \{\frac{\pi}{3}, \frac{5\pi}{3}\}$

(4) $y = x + \sin x$

let $y_1 = x$

$y_2 = \sin x$



(5) $\sin^{-1} : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos^{-1} : [-1,1] \to [0, \pi]$

$\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

(a) $\sin^{-1}(-1) = \frac{-\pi}{2}$ because $\sin\frac{-\pi}{2} = -1$ and $\frac{-\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(b) $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ b/c $\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$ and $\frac{5\pi}{6} \in [0, \pi]$

(c) $\tan^{-1}(-1) = \frac{-\pi}{4}$ b/c $\tan\left(\frac{-\pi}{4}\right) = -1$ and $\frac{-\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

(d) $\sec\left(\cos^{-1}\frac{1}{\sqrt{5}}\right) = \dfrac{1}{\cos\left(\cos^{-1}\frac{1}{\sqrt{5}}\right)} = \dfrac{1}{\frac{1}{\sqrt{5}}} = \sqrt{5}$

$\Big(\cos\left(\cos^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$ b/c

$f(f^{-1}(x)) = x$ if $x \in$ Domain f^{-1}

in our case, $\frac{1}{\sqrt{5}} \in [-1,1]$ $\Big)$

(e) $\cos^{-1}\left(\cos\frac{7\pi}{4}\right) \neq \frac{7\pi}{4}$

$\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$

note that $\frac{7\pi}{4} \notin [0, \pi]$

$\cos^{-1}\left(\cos\frac{7\pi}{4}\right) = \cos^{-1}\left(\cos\frac{\pi}{4}\right) = \frac{\pi}{4}$

(f) $\sin\left(\cos^{-1}\frac{1}{3}\right)$

let $\cos^{-1}\frac{1}{3} = u$, $u \in [0, \pi]$

then $\cos u = \frac{1}{3}$

$\sin^2 u + \cos^2 u = 1$

$\sin^2 u = 1 - \frac{1}{9} = \frac{8}{9}$

$\sin u = \mp \frac{2\sqrt{2}}{3}$

but $u \in [0, \pi]$, so $\sin u > 0$,

so $\sin u = \frac{2\sqrt{2}}{3}$

Therefore, $\sin\left(\cos^{-1}\frac{1}{3}\right) = \frac{2\sqrt{2}}{3}$

(6) (a) $\cot x - \tan x = \dfrac{\cos 2x}{\sin x \cos x}$

Proof

$$LHS = \cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{\cos 2x}{\sin x \cos x} = RHS$$

Therefore, the given equation is an identity

(b) $\dfrac{1}{1-\sin t} + \dfrac{1}{1+\sin t} = 2\sec^2 t$

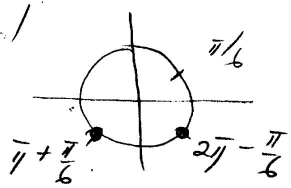
Proof

$$LHS = \frac{1}{1-\sin t} + \frac{1}{1+\sin t}$$
$$= \frac{1+\sin t + 1-\sin t}{(1-\sin t)(1+\sin t)}$$
$$= \frac{2}{1-\sin^2 t}$$
$$= \frac{2}{\cos^2 t}$$
$$= 2\sec^2 t = RHS$$

Therefore, the given equation is an identity

(7) $\sin 3a = \sin(a+2a)$
$$= \sin a \cos 2a + \sin 2a \cos a$$
$$= \sin a(1-2\sin^2 a) + 2\sin a\cos a\cos a$$
$$= \sin a - 2\sin^3 a + 2\sin a\cos^2 a$$
$$= \sin a - 2\sin^3 a + 2\sin a(1-\sin^2 a)$$
$$= \sin a - 2\sin^3 a + 2\sin a - 2\sin^3 a$$
$$= 3\sin a - 4\sin^3 a$$

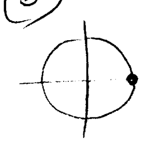
(8) (a) $2\sin 2x + 1 = 0$
$$2\sin 2x = -1$$
$$\sin 2x = \frac{-1}{2}$$



$$\begin{cases} 2x = \frac{7\pi}{6} + 2\pi k \\ \text{OR} \\ 2x = \frac{11\pi}{6} + 2\pi k \end{cases} \quad k \in \mathbb{Z}$$

$$\begin{cases} x = \frac{7\pi}{12} + \pi k \\ \text{OR} \\ x = \frac{11\pi}{12} + \pi k \end{cases}, \quad k \in \mathbb{Z}$$

(b) $\cos 4x = 1$ in $[0, 2\pi)$



$4x = 2\pi k, \quad k \in \mathbb{Z}$

$x = \frac{\pi k}{2}$

$k=0, \quad x=0$

$k=1, \quad x=\frac{\pi}{2}$

$k=2, \quad x=\pi$

$k=3, \quad x=\frac{3\pi}{2}$

$x \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$

(c) $3\tan\theta - 3 = 0$ in $[0, 2\pi)$
$$3\tan\theta = 3$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4} + \pi k, \quad k \in \mathbb{Z}$$

$k=0, \quad \theta = \frac{\pi}{4}$

$k=1, \quad \theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$\theta \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$

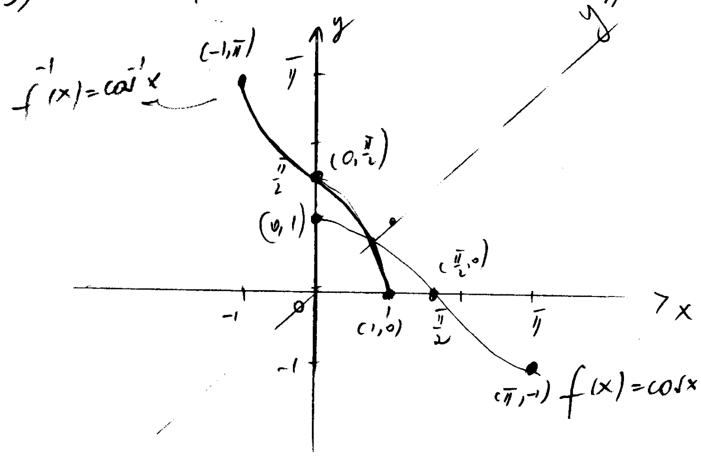
Math 150 TEST #2 - SOLUTIONS

(1) $f(x)=\cos x$

$f^{-1}(x)=\cos^{-1} x$

a) $\cos: [0,\pi] \to [-1,1]$

b) $\cos^{-1}: [-1,1] \to [0,\pi]$



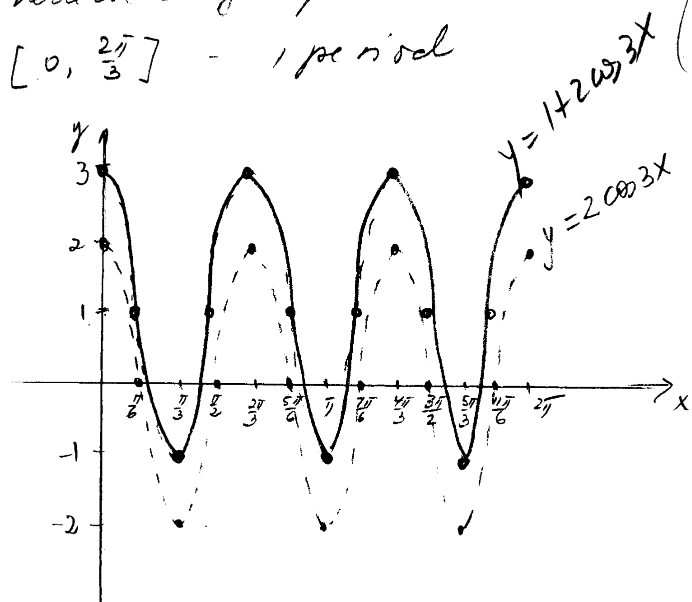
(2) (a) $y = 1+2\cos 3x$

$A=2$

$T=\frac{2\pi}{3}$

vertical shift up 1 unit

$[0, \frac{2\pi}{3}]$ - 1 period



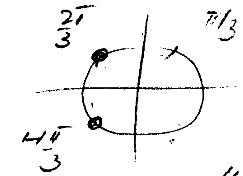
b) There are 6 x-intercepts.

x-int: let $y=0$

$1+2\cos 3x=0$

$2\cos 3x=-1$

$\cos 3x=-\frac{1}{2}$



$3x=\frac{2\pi}{3}+2\pi k$ OR $3x=\frac{4\pi}{3}+2\pi k$

$x=\frac{2\pi}{9}+\frac{2\pi k}{3}$ $\quad x=\frac{4\pi}{9}+\frac{2\pi k}{3}$

$k=0,\ x=\frac{2\pi}{9}$ or $x=\frac{4\pi}{9}$

$k=1,\ x=\frac{8\pi}{9}$ $\quad x=\frac{10\pi}{9}$

$k=2,\ x=\frac{14\pi}{9}$ $\quad x=\frac{16\pi}{9}$

$x\in\left\{\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{6}\right\}$

(3) $y=\frac{1}{2}\sin\left(2x+\frac{2\pi}{3}\right)$

$y=\frac{1}{2}\sin 2\left(x+\frac{\pi}{3}\right)$

$A=\frac{1}{2}$

$T=\frac{2\pi}{2}=\pi$

phase shift $=\frac{-\pi}{3}$

$[0,\pi] \to \left[\frac{-\pi}{3}, \pi-\frac{\pi}{3}\right]$

$\frac{-\pi}{3}$

$\frac{-\pi}{3}+\frac{\pi}{4}=\frac{-\pi}{12}$

$\frac{-\pi}{12}+\frac{\pi}{4}=\frac{2\pi}{12}$ $\quad \frac{\pi}{6}$

$\frac{2\pi}{12}+\frac{\pi}{4}=\frac{5\pi}{12}$

$\frac{5\pi}{12}+\frac{\pi}{4}=\frac{8\pi}{12}=\frac{2\pi}{3}$