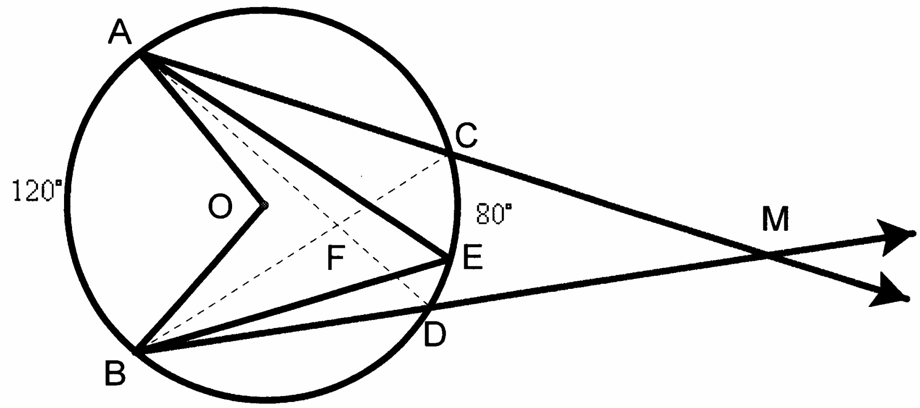
Math 61 Spring 2006 Name:

TEST #3 @ 130 points SOLUTIONS

Write in a neat and organized fashion. Use a straightedge and compass for your drawings.

1)



Given arcs: $m\widehat{AB}=120^\circ$ and $m\widehat{CD}=80^\circ$

Find:

a) $m\angle AOB = m\widehat{AB} = 120^\circ$

b) $m\angle CFD = \frac{1}{2}(m\widehat{AB} + m\widehat{CD}) = \frac{1}{2}(120^\circ + 80^\circ) = 100^\circ$

Name another angle that is congruent with $\angle CFD$: $\angle AFB$

c) $m\angle CBD = \frac{1}{2} m\widehat{CD} = \frac{1}{2}(80^\circ) = 40^\circ$

Name another angle that is congruent with $\angle CBD$: $\angle CAD$

d) $m\angle AEB = \frac{1}{2} m\widehat{AB} = \frac{1}{2}(120^\circ) = 60^\circ$

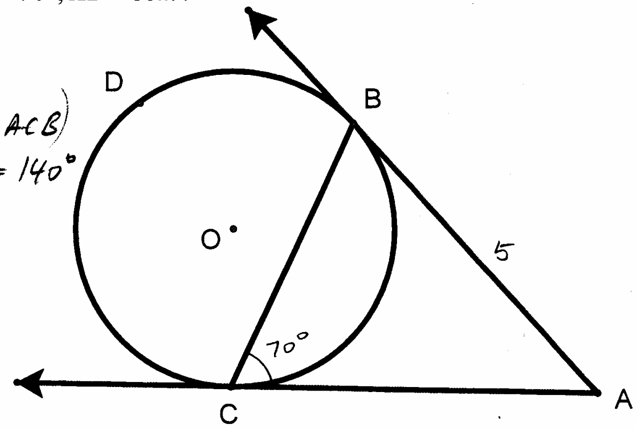
Name two other angles that are congruent with $\angle AEB$: $\angle ADB$ and $\angle ACB$

e) $m\angle AMB = \frac{1}{2}(m\widehat{AB} - m\widehat{CD})$

$= \frac{1}{2}(120^\circ - 80^\circ)$

$= \frac{1}{2}(40^\circ) = 20^\circ$

2) Given: $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are tangents to $\odot O$, $m\angle ACB = 70^\circ$, $AB = 5cm$.



Find:

a) $m\widehat{BC}$

$m\angle ACB = \frac{1}{2} m\widehat{BC} \Rightarrow m\widehat{BC} = 2m(\angle ACB) = 2(70^\circ) = 140^\circ$

b) $m\widehat{BDC} = 360^\circ - m(\widehat{BC})$
$= 360^\circ - 140^\circ$
$= 220^\circ$

c) $m\angle ABC = \frac{1}{2} m\widehat{BC}$
$= \frac{1}{2}(140^\circ) = 70^\circ$

d) $m\angle A = \frac{1}{2}(m\widehat{BDC} - \widehat{BC})$
$= \frac{1}{2}(220^\circ - 140^\circ) = 40^\circ$

(OR, in $\triangle ABC$: $m\angle A = 180^\circ - m\angle B - m\angle C$)

e) $AC = AB = 5\,cm$ because $\overline{AB}$ and $\overline{AC}$ are tangent segments and $\overline{AB} \cong \overline{AC}$ (tangents to $\odot$ $\cong$)

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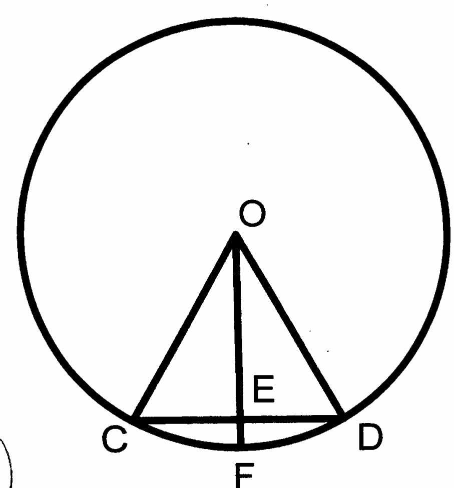
3) Given: $\odot O$ with $\overline{OE} \perp \overline{CD}$
$CD = OC$



Find: $m\widehat{CF}$

Solution

$\triangle OCD$ = equilateral ($\overline{OC} \cong \overline{OD}$ radii; $\overline{OC} \cong \overline{CD}$ given)

$\Rightarrow m\angle COD = 60^\circ$

$\overline{OE} \perp \overline{CD} \Rightarrow \overline{OE}$ bisects $\overline{CD}$ and $\widehat{CD}$
(sec thru center $\perp$ chord bisects chord and arc)

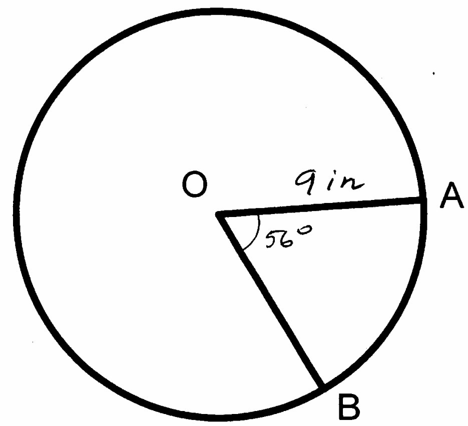
$\Rightarrow m\widehat{CF} = m\widehat{FD}$
But $m\widehat{CD} = m\angle COD = 60^\circ$
$\Rightarrow m\widehat{CF} = \frac{1}{2}(60^\circ) = 30^\circ$

$\boxed{m\widehat{CF} = 30^\circ}$

4) Given $\odot O$ with $m\angle AOB = 56^\circ$

$OA = 9in$

$\pi \approx \frac{22}{7}$



Find and use correct units:

a) $m\widehat{AB}$ $= m(\angle AOB) = 56^\circ$

b) $l\widehat{AB}$

$$\frac{l\widehat{AB}}{56^\circ} = \frac{2\pi r}{360^\circ} \Rightarrow l\widehat{AB} = \frac{2\pi r \cdot 56^\circ}{360^\circ}$$

$$l\widehat{AB} \approx \frac{2 \cdot 22 \cdot \cancel{9} \cdot \cancel{56}^{8}}{\cancel{7} \cdot \cancel{360}_{40}}\ in = \frac{2 \cdot 22 \cdot \cancel{8}^{1}}{\cancel{40}_{5}} = \frac{44}{5} = 8.8\ in$$

c) Circumference of the circle

$$C = 2\pi r = 2 \cdot \frac{22}{7} \cdot 9\ in = \frac{396}{7}\ in \approx 56.6\ in$$

d) Area of the circle

$$A = \pi r^2 = \frac{22}{7} \cdot 9^2 = \frac{1782}{7}\ in^2 \approx 254.6\ in^2$$

e) Area of the sector AOB

$$\frac{A_D(AOB)}{56^\circ} = \frac{A_O}{360^\circ} \Rightarrow A_D(AOB) = \frac{\pi r^2 \cdot 56^\circ}{360^\circ}$$

$$A_D(AOB) = \frac{22 \cdot 9^{\cancel{2}} \cdot \cancel{56}^{8}}{\cancel{7} \cdot \cancel{360}_{40\ 5}}\ in^2 = \frac{22 \cdot 9}{5}\ in^2 = \frac{198}{5}\ in^2 \approx 39.6\ in^2$$

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5) Simplify the following trigonometric expressions:

a) $\frac{\sec^2 x - 1}{\sec^2 x} =$

$$= \frac{\frac{1}{\cos^2 x} - 1}{\frac{1}{\cos^2 x}} = \frac{\frac{1-\cos^2 x}{\cancel{\cos^2 x}}}{\frac{1}{\cancel{\cos^2 x}}}$$

$$= \frac{1-\cos^2 x}{1} = \boxed{\sin^2 x}$$

(b/c $\sin^2 x + \cos^2 x = 1$)

b) $\cos^3\theta + \sin^2\theta\cos\theta =$

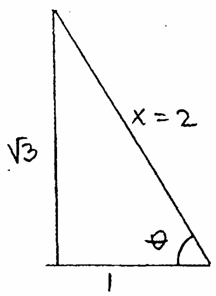
$$= \cos\theta(\cos^2\theta + \sin^2\theta)$$

$$= (\cos\theta)(1)$$

$$= \boxed{\cos\theta}$$

6) Sketch a right triangle that has one acute angle θ, and find the other five trigonometric ratios of θ knowing that

$\tan\theta = \sqrt{3}$



$\tan\theta = \frac{opp}{adj} \Rightarrow opp = \sqrt{3}$ and $adj = 1$

Then, if x = hypotenuse, $x^2 = 1^2 + (\sqrt{3})^2 = 4$

$x = 2$

$\sin\theta = \frac{\sqrt{3}}{2}$

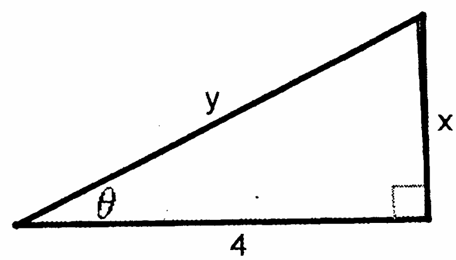
$\cos\theta = \frac{1}{2}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{2}{1}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

7) Express x and y in terms of trigonometric ratios of θ



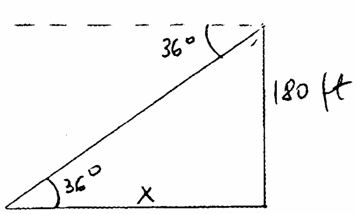
$x = ?$ $\quad \tan\theta = \frac{x}{4} \Rightarrow x = 4\tan\theta$

$y = ?$ $\quad \cos\theta = \frac{4}{y} \Rightarrow y = \frac{4}{\cos\theta}$

OR

$y = 4\sec\theta$

8) From the top of a 180-ft lighthouse, the angle of depression to a ship in the ocean is $36°$. How far is the ship form the base of the lighthouse?



Let x = distance between ship and base of lighthouse

$\tan 36° = \frac{180\text{ ft}}{x} \Rightarrow x = \frac{180\text{ ft}}{\tan 36°}$

$x \approx 248$ ft

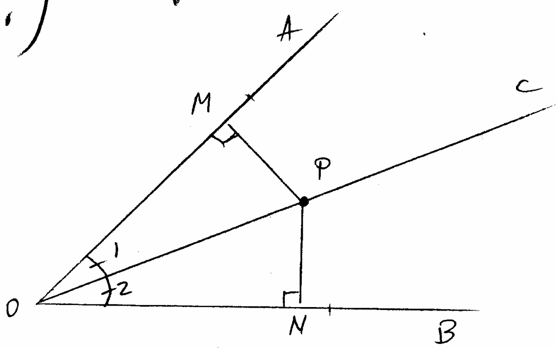
9) Prove ONE of the following locus problems. Do not prove both.

I) The locus of points in a plane equidistant from the sides of an angle is the angle bisector.

II) The locus of points in a plane that are equidistant from the endpoints of a line segment is the perpendicular bisector of that line segment.

(I)

(i) We will show that any point in the locus (angle bisector) is equidistant from the sides of an angle
and
(ii) Any point equidistant from the sides of an angle is in the locus (on the angle bisector)



(i) Given $\angle AOB$

$\overrightarrow{OC}$ bisector of $\angle AOB$

$P \in \overrightarrow{OC}$

Prove $d(P, \overrightarrow{OA}) = d(P, \overrightarrow{OB})$

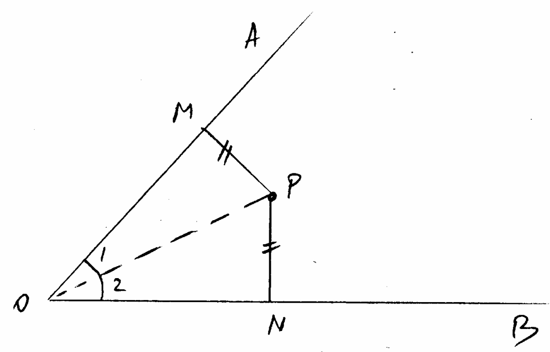
Draw $\overline{PM} \perp \overrightarrow{OA}$, $M \in \overrightarrow{OA}$

$\overline{PN} \perp \overrightarrow{OB}$, $N \in \overrightarrow{OB}$

We'll show $\overline{PM} \cong \overline{PN}$

$\triangle MOP$, $\triangle NOP$:
$\angle 1 \cong \angle 2$ (b/c $\overrightarrow{OP}$ bisector)
$\overline{OP} \cong \overline{OP}$ ($\overline{OP}$ common side) . reflexive . prop
$\angle OMP \cong \angle ONP$ (as right $\angle$s)

$\Rightarrow \triangle MOP \cong \triangle NOP$ (AAS) $\Rightarrow$ $\overline{MP} \cong \overline{NP}$



(ii) Given $\angle AOB$

$P \in int \angle AOB$

$d(P, \overrightarrow{OA}) = d(P, \overrightarrow{OB})$

Prove $P \in$ bisector of $\measuredangle AOB$

Draw $\overline{OP}$

We know $\overline{MP} \cong \overline{NP}$ ($\overline{MP} = d(P, \overrightarrow{OA})$, $\overline{NP} = d(P, \overrightarrow{OB})$)

We'll show $\angle 1 \cong \angle 2$ ($\overline{OP}$ = bisector)

$\triangle MOP$ $\triangle NOP$ right $\triangle$s $\left\{ \begin{array}{ll} \overline{OP} \cong \overline{OP} & \text{(common hypotenuse) - reflexive prop.} \\ \overline{MP} \cong \overline{NP} & \text{(given)} \end{array} \right.$

$\Rightarrow \triangle MOP \cong \triangle NOP$ (HL) $\Rightarrow \angle 1 \cong \angle 2$

$\Rightarrow \overrightarrow{OP}$ = bisector $\angle AOB$

q.e.d.

(II) See textbook

Theorem 6.5.2 on page 293–294

EXTRA CREDIT

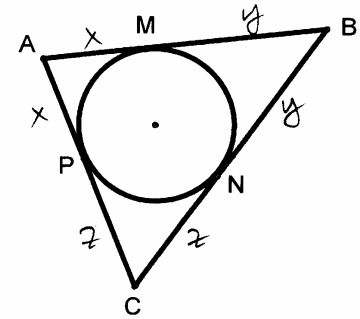
Choose ONE or TWO of the following problems:

(1) @ 10 points

A water tower 30 m tall is at the top of a hill. From a distance of 120 m down the hill it is observed that the angle formed between the top and the base of the tower is 8°. Find the angle of inclination of the hill.

(2) @ 5 points

A circle is inscribed in the triangle ABC as shown.
$AB = 14, BC = 16, AC = 12$.
Find AM, PC, BN.



(3) @ 5 points

Verify the identity: $\tan^2 x - \cot^2 x = \sec^2 x - \csc^2 x$

(3) $\tan^2 x - \cot^2 x = \dfrac{\sin^2 x}{\cos^2 x} - \dfrac{\cos^2 x}{\sin^2 x}$

$= \dfrac{\sin^4 x - \cos^4 x}{\sin^2 x \cos^2 x} = \dfrac{(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}$

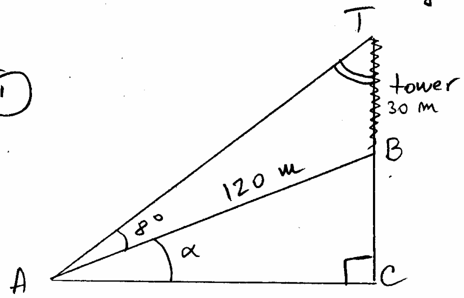
$= \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}$

$= \dfrac{\sin^2 x}{\sin^2 x \cos^2 x} - \dfrac{\cos^2 x}{\sin^2 x \cos^2 x}$

$= \dfrac{1}{\cos^2 x} - \dfrac{1}{\sin^2 x}$

$= \sec^2 x - \csc^2 x$

①



Let $\alpha = \measuredangle$ of inclination of the hill

ΔTAB - Law of sines

$$\frac{30\text{ m}}{\sin 8^\circ} = \frac{120\text{ m}}{\sin \measuredangle T} \Rightarrow$$

$$\sin \measuredangle T = \frac{(120\text{ m})(\sin 8^\circ)}{30\text{ m}} = 4 \sin 8^\circ \approx 0.557$$

$\Rightarrow \measuredangle T = \sin^{-1}(0.557) = 34^\circ$
(or using the table on page 465)

ΔTAC: $\quad 8^\circ + \alpha + 34^\circ = 90^\circ$

$\alpha = 90^\circ - 42^\circ$

$\alpha = 48^\circ$

② $AM = AP = x$ (tangents to ⊙ ≅)

$BM = BN = y$ — " —

$CP = CN = z$ — " —

Then, $AB = 14 \iff x + y = 14$

$AC = 12 \iff x + z = 12$

$BC = 16 \iff y + z = 16$

$$\begin{cases} x + y = 14 & ① \\ x + z = 12 & ② \\ y + z = 16 & ③ \end{cases}$$

$① - ③ \Longrightarrow x - z = -2 \quad ④$

but $x + z = 12 \quad ②$

$(+) \quad 2x = 10 \Rightarrow$

$x = 5$

② $\Rightarrow z = 7$

① $\Rightarrow y = 9$

Therefore, $\boxed{\begin{array}{l} AM = 5 \\ PC = 7 \\ BN = 9 \end{array}}$