Section 7.2
Exponential Functions
Exponential Growth and Decay

The story of chess . It seems that chess had its beginnings in India around 600AD -1,400 years ago. There was a King in India who loved to play games. He commissioned a poor mathematician who lived in his kingdom to come up with a new game. After months of struggling with all kinds of ideas the mathematician came up with the game of Chaturanga. The game had two armies each lead by a King who commanded the army to defeat the other by capturing the enemy King. It was played on a simple 8x8 square board. The King loved this game so much that he offered to give the mathematician anything he wished for. ***"I would like one grain of rice for the first square of the board, two grains for the second, four grains for the third and so on doubled for each of the 64 squares of the game board"*** said the mathematician. "Is that all?" asked the King, "Why don't you ask for gold or silver coins instead of rice grains". "The rice should be sufficient for me." replied the mathematician. The King ordered his staff to lay down the grains of rice and soon learned that all the wealth in his kingdom would not be enough to buy the amount of rice needed on the 64th square. In fact the whole kingdoms supply of rice was exhausted before the 40th square was reached. "You have provided me with such a great game and yet I cannot fulfill your simple wish. You are indeed a genius." said the King and offered to make the mathematician his top most advisor instead.

Question #1: Can you find exactly how many grains of rice would be needed on the 64th square and how much total rice would be needed for all 64 squares?

9,223,372,036,854,775,808 on the 64^{th} square and 18,446,744,073,709,551,615 total for the whole board. That's about 18 billion billions. So if a bag of rice contained a billion grains, you would need 18 billion such bags.

Question #2: Suppose that your mathematics instructor, in an effort to improve classroom attendance, offers to pay you each day for attending class! Suppose you are to receive 2 cents on the first day you attend class, 4 cents the second day, 8 cents the third day, and so on for 30 days. What would you rather have: $1 million dollars or the above offer?

Note: Simple method for quickly estimating powers of two

$$2^{10} \approx 10^3$$

Review

Complete the following properties:

$b^m b^n =$	$(ab)^n =$
$\frac{b^m}{b^n} =$	$\left(\frac{\text{a}}{\text{b}}\right)^n =$
$\left(b^m\right)^n =$	$b^{\frac{m}{n}} =$
$b^0 =$	
$b^{-n} =$	

1) Write using radical notation:

a) $10^{\frac{4}{5}}$ b) $x^{\frac{3}{7}}$

2) Use exponent notation:

a) $\sqrt[7]{(1+n)^4}$ b) $\sqrt[3]{x^5}$

3) Simplify :

a) $16^{\frac{1}{2}}$ b) $(-32)^{\frac{1}{5}}$ c) $64^{-\frac{1}{2}}$

4) If $f(x) = 3^x$, find each of the following:

a) $f(2)$ b) $f(-3)$

5) Solve the following equations:

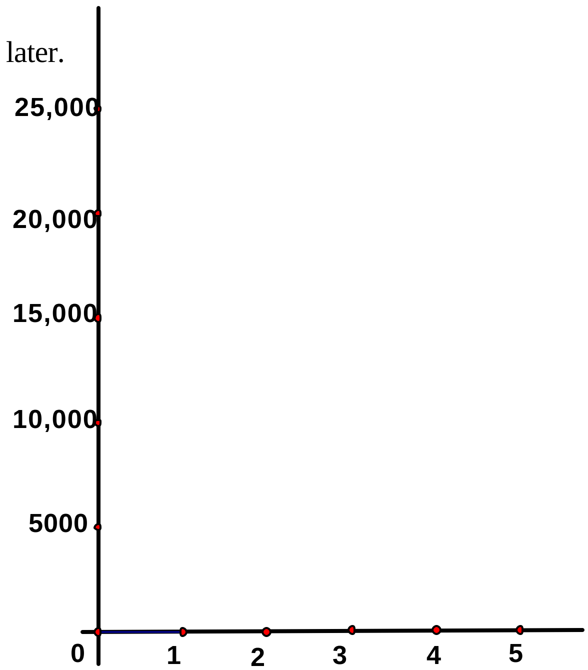
a) $3^x = 27$ b) $2^{3y+1} = \sqrt{2}$ c) $\left(\frac{1}{2}\right)^k = 4$

Exponential Growth and Decay

Example 1 In a laboratory experiment, researchers establish a colony of 100 bacteria and monitor its growth. The experiments discover that the colony triples in population every day.

1. Fill in Table 1. showing the population, $P(t)$, of bacteria t days later.

t	P(t)
0	
1	
2	
3	
4	
5	



2. Plot the data points from Table 1. and connect them with a smooth curve.

3. Write a function that gives the population of the colony at any time t in days. (Express the values you calculated in part (1) using powers of 3. Do you see a connection between the value of t and the exponent on 3?)

4. Evaluate your function to find the number of bacteria present after 8 days. How many bacteria are present after 36 hours?

Growth Factors

The function in Example 1 describes **exponential growth.** During each time interval of a fixed length the population is multiplied by a certain constant amount. The bacteria population grows by a factor of 3 every day. For this reason we say that 3 is the **growth factor** for the function.

Functions that describe exponential growth can be expressed in the standard form

$$A(t) = A_0 b^t$$

or

$$A(t) = A_0 e^{kt}$$

$A_0 = A(0)$ is the initial value of the function
a is the growth factor

$A_0 = A(0)$ is the initial value of the function
k is the relative growth rate (k>0)

Example 2 **<u>Percent Increase</u>**

Exponential growth occurs in other circumstances, too. For example, of the interest on a savings account is compounded annually, the amount of money in the account grows exponentially.

Consider a principal of \$100 invested at 5% interest compounded annually.

What is the amount in the account at the end of 1 year?
Write the formula for the amount in factored form.

t	A(t)
0	
1	
2	
3	

The amount, \$105, becomes the new principal for the second year.
To find the amount at the end of the second year, we apply the formula again, with $P = 105$.
Observe that to find the amount at the end of each year we multiply the principal by a factor of $1 + r = 1.05$.

A formula for the amount after *t* years is $A(t) = 100(1.05)^t$.

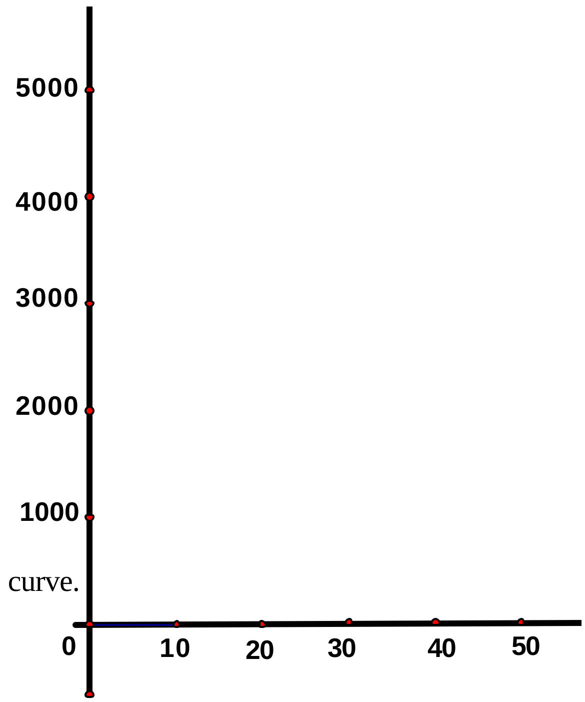
In general, for an initial investment of *P* dollars at an interest rate , *r*, <u>compounded annually</u>, the amount accumulated after *t* years is

$$A(t) = P(1 + r)^t$$

Example 3 A small coal-mining town has been losing population since 1940, when 5000 people lived there. At each census thereafter (taken at 10-year intervals) the population has declined to approximately 0.90 of its earlier figure.

1. Fill in Table 2. showing the population, $P(t)$, of the town t years after 1940.

t	P(t)
0	
10	
20	
30	
40	
50	



2. Plot the data points from Table 2. and connect them with a smooth curve.

3. Write a function that gives the population of the town at any time t in years after 1940. (Express the values you calculated in part (1) using powers of 0.90. Do you see a connection between the value of t and the exponent on 0.90?)

4. Evaluate your function to find the population of the town in 1995. What was the population in 2000?

Exponential Growth and Decay Functions

The function

$$A(t) = A_0 e^{kt} \quad (k > 0) \text{ models exponential growth.}$$

The function

$$A(t) = A_0 e^{-kt} \quad (k > 0) \text{ models exponential decay.}$$

$A_0 = A(0)$ is the **initial value** of A

k is the relative growth or decay rate

.

<u>The Exponential Function</u>: $f(x) = ab^x$, where $b > 0$ and $b \neq 1, a \neq 0$.

a = the coefficient, b = base.

<u>Note:</u> If $b = 1$, then $b^x = 1^x = 1$ for any x – trivial.

If $b = 0$, then $b^x = 0^x$ which is undefined when $x = 0$

If $b < 0$, let $f(x) = (-2)^x$. If $x = \frac{1}{2}$, $f(x) = (-2)^{\frac{1}{2}} = \sqrt{-2} \notin \mathbb{R}$

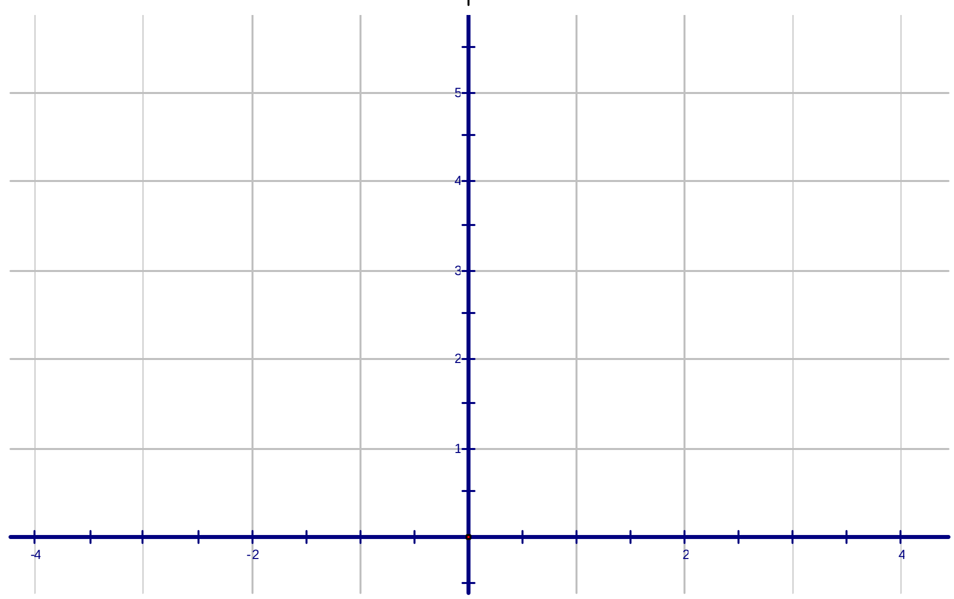
<u>Graphs of Exponential Functions</u>

$$f(x) = 2^x$$

x	$-\infty$ ∞
y	

$$g(x) = \left(\frac{1}{2}\right)^x$$

x	$-\infty$ ∞
y	



Domain: ______________

Range: ______________

Horizontal Asymptote: ______________

If $b > 1$, the function is **increasing**.

Domain: ______________

Range: ______________

Horizontal Asymptote: ______________

If $0 < b < 1$, the function is **decreasing.**

Question: Which function grows more rapidly: $y = 3^x$ or $y = 4^x$?

When $b > 1$, the greater the value of b is, the more ______________________

Which function decreases more rapidly: $y = (0.5)^x$ or $y = (0.8)^x$?

When $0 < b < 1$, the smaller the value of b is, the more ______________________

Compound Interest

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A= amount in the account after *t* years
P = principal (amount invested)
r = annual interest rate
n = number of times interest is compounded per year
t = number of years

Example 4 Assume we invest $10,000 in an account that pays 6% interest rate per year. How much is in the account at the end of one year if

a) interest is compounded quarterly?

b) interest is compounded monthly?

The Number *e*

An interesting situation occurs if we consider the compound interest formula for $P = \$1$, $r = 100\%$, $t = 1$ year.

The formula becomes ______________________________.

The following table shows some values, rounded to eight decimal places, of $\left(1+\frac{1}{n}\right)^{n}$

n	$\left(1+\frac{1}{n}\right)^{n}$
1	2.00000000
10	2.5937246
100	2.70481383
1000	2.71692393
10,000	2.71814593
100,000	2.71826824
1,000,000	2.71828047
10,000,000	2.71828169
100,000,000	2.71828181
1,000,000,000	2.71828183

a) For a fixed period of time (say one year), does more and more frequent compounding of interest continue to yield greater and greater amounts?

b) Is there a limit on how much money can accumulate in a year when interest is compounded more and more frequently?

The table suggests that as n increases, the value of $\left(1+\frac{1}{n}\right)^{n}$ gets closer and closer to some fixed number. The fixed number is called *e*. To five decimal places, *e* = 2.71828 .

When $n \to \infty$, $\left(1+\frac{1}{n}\right)^{n} \to e$ and the formula for

continuously compounded interest is $$A = Pe^{rt}$$

Example 5 Assume we invest \$10,000 in an account that pays 6% interest rate per year compounded continuously.

a) How much is in the account at the end of one year?
b) How much interest was paid in one year under the continuous compounding?
c) How much is in the account at the end of ten years?

Exercise #1 India is currently one of the world's fastest-growing countries. By 2040, the population of India will be larger than the population of China; by 2050, nearly one-third of the world's population will live in these two countries alone. The exponential function $f(x) = 574(1.026)^x$ models the population of India, $f(x)$, in millions, x years after 1974.

a) What was India's population in 1974?

b) Find $f(27)$ and its meaning.

c) Find India's population, to the nearest million, in the year 2028 as predicted by this function.

Exercise #2 (7.2 - #39) A population has an initial size of 3,241 with a 2.5% annual relative growth rate. Estimate the population size in ten years.

Exercise #3 (7.2 - # 42) A radioactive substance has an initial mass of 25.9 g and an yearly relative decay rate of 0.15%. Give the amount of substance that remains in 33 years. How many ounces is this?