Section 3.4 – The Derivative as a Rate of Change
Rates of Change in the Natural and Social Sciences

Physics

If $s = f(t)$ is the position function of a particle that is moving in a straight line, then $\frac{\Delta s}{\Delta t}$ represents the **average velocity over a time period** Δt, and $v = \frac{ds}{dt} = s'$ represents the **instantaneous velocity** (the rate of change of displacement with respect to time. The **acceleration is** $a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$. A sudden change in acceleration is called a **jerk**. Jerk is the derivative of acceleration with respect to time; $j = \frac{da}{dt} = \frac{d^3 s}{dt^3}$.

EXAMPLE 3 Horizontal Motion

Figure 3.16 shows the velocity $v = f'(t)$ of a particle moving on a coordinate line. The particle moves forward for the first 3 sec, moves backward for the next 2 sec, stands still for a second, and moves forward again. The particle achieves its greatest speed at time $t = 4$, while moving backward. ■

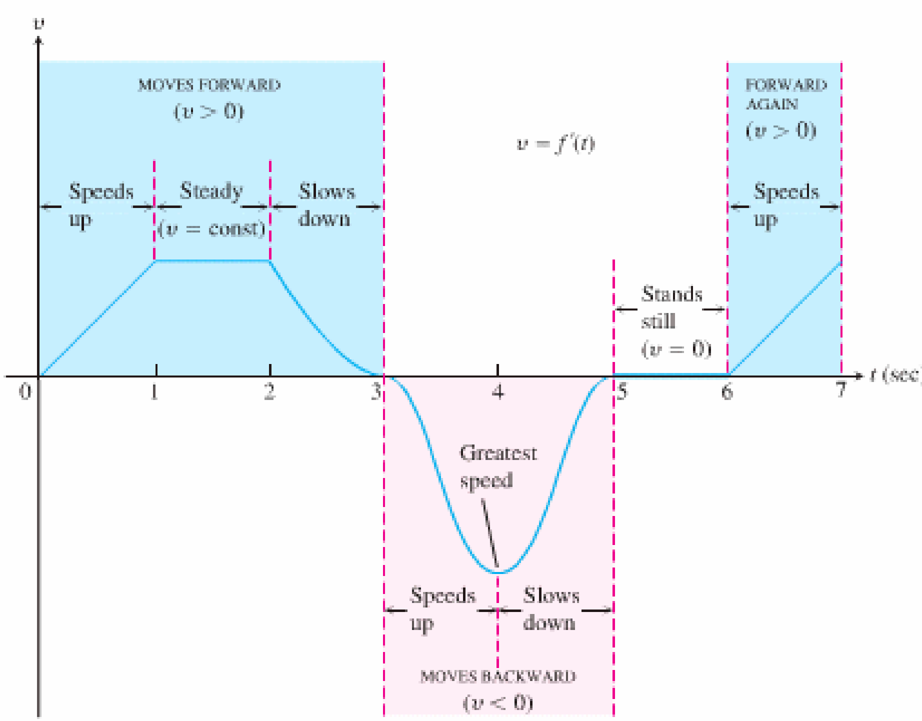


FIGURE 3.16 The velocity graph for Example 3.

Exercise 1 The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters

a) Find the velocity at time *t*.
b) What is the velocity after 2 s? After 4 s?
c) When is the particle at rest?
d) When is the particle moving forward (that is, in the positive direction)? When is it moving backward?
e) Find the total distance traveled by the particle during the first five seconds.
f) Find the acceleration at time t and after 4 s.
g) When is the particle speeding up? When is it slowing down?

Exercise 2 A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec. It reaches a (3.4:Example 4) height of $s = 160t - 16t^2$ ft after t sec.

a) How high does the rock go?
b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
c) What is the acceleration of the rock at any time t during its flight (after the blast)?
d) When does the rock hit the ground again?

Economics

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called the **cost function**. If the number of items produced is increased from x_1 to x_2, then the **additional cost** is $\Delta C = C(x_2) - C(x_1)$, and the **average rate of change of the cost** is $\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$. The instantaneous rate of change of cost with respect to the number of items produces is called the **marginal cost** and it is $\frac{dC}{dx} = C'$.

It is often appropriate to represent a total cost function by a polynomial $C(x) = a + bx + cx^2 + dx^3$, where a represents overhead cost (rent, heat, maintenance) and the other terms represent cost of raw materials, labor, and so on. (The cost of raw materials may be proportional to *x*, but labor costs might depend partly on higher powers of *x* because of overtime costs and inefficiencies involved in large-scale operations.)

For instance, suppose a company has estimated that the cost (in dollars) of producing x items is

$$C(x) = 10{,}000 + 5x + 0.01x^2.$$

Then the marginal cost function is

$$C'(x) = 5 + 0.02x$$

The marginal cost at the production level of 500 items is

$$C'(500) = 5 + 0.02(500) = \$15/item$$

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st item.

The actual cost of producing the 501st item is

$$C(501) - C(500) = [10{,}000 + 5(501) + 0.01(501)^2] - [10{,}000 + 5(500) + 0.01(500)^2] = \$15.01$$

Notice that $C'(500) \approx C(501) - C(500)$.

Economists also study marginal demand, marginal revenue, and marginal profit, which are the derivatives of the demand, revenue, and profit functions.

Example 1 The cost C (in dollars) of building a house A square feet in area is given by the function $C = f(A)$. What is the practical interpretation of the function $f'(A)$?

Solution

In the alternative notation,

$$f'(A) = \frac{dC}{dA}.$$

This is the cost divided by an area, so it is measured in dollars per square foot. You can think of dC as the extra cost of building an extra dA square feet of house. Thus, dC/dA is the additional cost per square foot. So if you are planning to build a house roughly A square feet in area, $f'(A)$ is the cost per square foot of *extra* area involved in building a slightly larger house, and it is called the *marginal cost*. The marginal cost is not necessarily the same thing as the average cost per square foot for the entire house, since once you are already set up to build a large house, the cost of adding a few square feet could be comparatively small.

Example 2 The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

Solution

In the alternative notation, $f'(2000) = \left.\frac{dC}{dT}\right|_{T=2000}$. Since C is measured in dollars and T is measured in tons, dC/dT must be measured in dollars per ton. So the statement $\left.\frac{dC}{dT}\right|_{T=2000} = 100$ says that when 2000 tons of ore have already been extracted from the mine, the cost of extracting the next ton is approximately \$100. In other words, after 2000 tons have been removed, extraction costs are \$100 per ton. Another way of saying this is that it costs about \$100 to extract ton number 2000 or 2001. Note that this may well be different from the cost of extracting the tenth ton, which is likely to be more accessible.

Example 3 You are told that water is flowing through a pipe at a rate of 10 cubic feet per second. Interpret this rate as the derivative of some function.

Solution

You might think at first that the statement has something to do with the velocity of the water, but in fact a flow rate of 10 cubic feet per second could be achieved either with a very slowly moving water through a large pipe, or with very rapidly moving water through a narrow pipe. If we look at the units – cubic feet per second – we realize that we are being given the rate of change of a quantity measured in cubic feet. But a cubic foot is a measure of volume, so we are being told the rate of change of a volume. If you imagine all the water that is flowing through ending up in a tank somewhere and let $V(t)$ be the volume of the tank at time t, then we are being told that the rate of change of $V(t)$ is 10, or

$$V'(t) = \frac{dV}{dt} = 10$$

Exercise 3 (3.4: Example 5)

Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

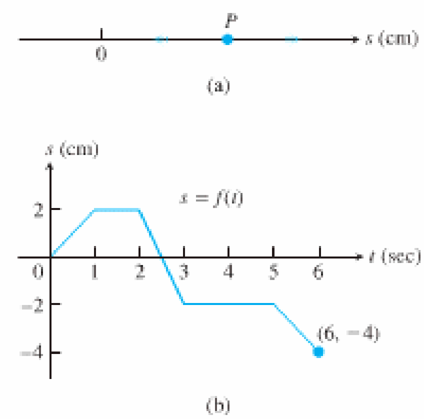
gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. How much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Exercise 4 (3.4 - # 16)

A particle *P* moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of *P* as a function of time *t*.

a) When is *P* moving to the left? Moving to the right? Standing still?

b) Graph the particle's velocity and speed (where defined).



Exercise 5 (3.4 - #23)

Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a) Find the average cost per machine of producing the first 100 washing machines.

b) Find the marginal cost when 100 washing machines are produced.

Exercise 6

In the study of ecosystems, predator-prey models are often used to study the interaction between species. Consider a population of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

a) What values of dC/dt and dW/dt correspond to stable populations?
b) How would the statement "The caribou go extinct" be represented mathematically?
c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the species to live in harmony or will one or both species become extinct?

Section 3.10 – Related Rates

Exercise 1
(Example 1 – 3.10)

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out out at a rate of 3000 L/min?

Exercise 2
(Example 3 – 3.10)

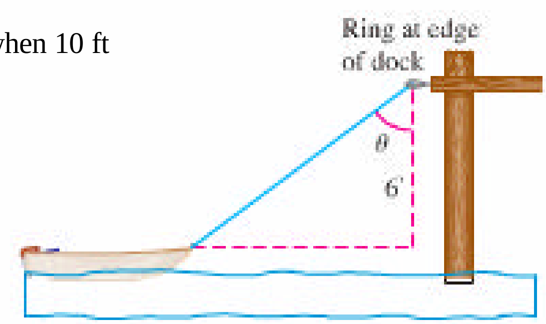
A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Exercise 3
(3.10 - # 32)

A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at a rate of 2 ft/sec.

a) How fast is the boat approaching the dock when 10 ft of rope are out?

b) At what rate is the angle q changing then?



Exercise 4
(3.10 - #36)

A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination q of the line joining the particle to the origin changing when $x = 3$ m?

Exercises 5
t(3.10 - # 41)

A spherical iron ball 8 in in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 in^3 /\text{min}$, how fast is the thickness of the ice decreasing when it is 2 inches thick? How fast is the outer surface area of ice decreasing?

Exercise 6
(3.10 - #21)

The length *l* of a rectangle is decreasing at the rate of 2 cm/sec while the width *w* is increasing at the rate of 2 cm/sec. When *l*=12 cm and *w*=5 cm, find the rates of change of (a) the area, (b) the perimeter, and (c) the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

Exercises 7
(3.10 - #30)

Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop/s radius increases at a constant rate.