HANDOUT 5.3 — SOLUTIONS

EXERCISE 1 - page 3

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left[c_k^3 + c_k \sin c_k\right]\Delta x = \int_0^{\pi}\left(x^3 + x\sin x\right)dx$$

EXERCISE 2 - page 3

$f(x) = x^3 - 6x$

$[a,b] = [0,3]$, $n = 6$

(a) $R_6 = \sum_{k=1}^{6} f(c_k)\,\Delta x$, where

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

$c_0 = 0$, $c_1 = 0.5$, $c_2 = 1$, $c_3 = 1.5$, $c_4 = 2$, $c_5 = 2.5$, $c_6 = 3$ (in general, $c_k = \frac{k}{2}$)

$$R_6 = \sum_{k=1}^{6}\left(c_k^3 - 6c_k\right)\frac{1}{2} =$$

$$= \frac{1}{2}\sum_{k=1}^{6} c_k^3 - 6\cdot\frac{1}{2}\sum_{k=1}^{6} c_k$$

$$= \frac{1}{2}\sum_{k=1}^{6}\left(\frac{k}{2}\right)^3 - 3\sum_{k=1}^{6}\left(\frac{k}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{8}\sum_{k=1}^{6} k^3 - \frac{3}{2}\sum_{k=1}^{6} k$$

$$= \frac{1}{16}\cdot\frac{6^2(6+1)^2}{4} - \frac{3}{2}\,\frac{6(6+1)}{2}$$

$$= -3.9375$$

$$R_6 = -3.9375$$

Note that R_6 does not represent a sum of areas

$R_6 \approx A_1 - A_2$, where A_1 = area above x-axis, A_2 = below

(b) $$\int_0^3 \left(x^3 - 6x\right)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta x$$

where $\Delta x = \frac{b-a}{n} = \frac{3}{n}$

choose $x_k = k\cdot\frac{3}{n}$

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\frac{3k}{n}\right)^3 - 6\cdot\frac{3k}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty}\frac{3}{n}\left(\frac{27}{n^3}\sum_{k=1}^{n} k^3 - \frac{18}{n}\sum_{k=1}^{n} k\right)$$

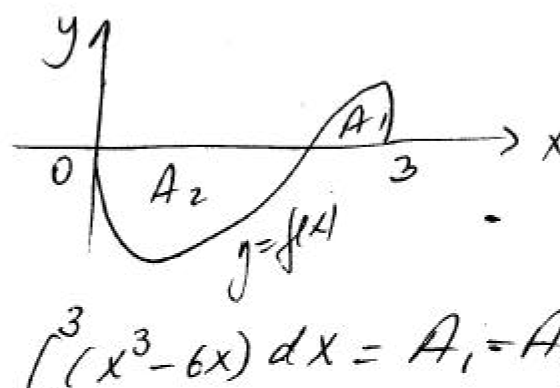
$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left(\frac{81}{4}\cdot\frac{(n+1)^2}{n^2} - 27\,\frac{n(n+1)}{n^2}\right)$$

$$= \frac{81}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 - 27\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{81}{4} - 27 = \frac{-27}{4} = -6.75$$

Note that this integral cannot be interpreted as an area because it takes on both + and − values.



$$\int_0^3 \left(x^3 - 6x\right)dx = A_1 - A_2$$

EXERCISE 3 - page 4

(a) $\int_0^1 \sqrt{1-x^2}\, dx = ?$

let $f(x) = \sqrt{1-x^2}$

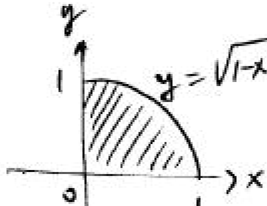
note that $f(x) \geq 0$

so $\int_0^1 f(x)\, dx$ = area under $y = \sqrt{1-x^2}$ from 0 to 1 and the x-axis

$y = \sqrt{1-x2}$

$y^2 = 1-x^2$

$x^2+y^2 = 1$ circle of center $(0,0)$ and radius 1



so $y = \sqrt{1-x^2}$ is the upper semicircle

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi = \frac{\pi}{4}$$

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$$

(b) $\int_0^3 (x-1)\, dx = ?$

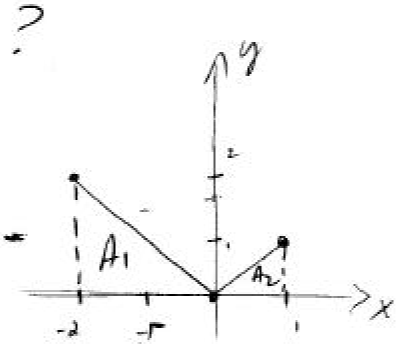
let $f(x) = x-1$

$$\int_0^3 (x-1)\, dx = A_1 - A_2 = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1) = 1.5$$

$$\int_0^3 (x-1)\, dx = 1.5$$

(c) $\int_{-2}^1 |x|\, dx = ?$

let $f(x) = |x|$



$$\int_{-2}^1 |x|\, dx = A_1 + A_2 = \frac{1}{2}(2\cdot 2) + \frac{1}{2}(1\cdot 1) = 2.5$$

EXERCISE 4 - page 4

(a) $\int_0^1 (4+3x^2)\, dx = \int_0^1 4\, dx + \int_0^1 3x^2\, dx$

$= \int_0^1 4\, dx + 3\int_0^1 x^2\, dx$

$= 4(1-0) + \frac{1}{3}$ (proved in 5.1 Handout)

so $\int_0^1 (4+3x^2)\, dx = 5$

(b) $\int_0^{10} f(x)\, dx = \int_0^8 f(x)\, dx + \int_8^{10} f(x)\, dx$

$\Rightarrow \int_8^{10} f(x)\, dx = 17 - 12 = 5$

(c) we want to find m, M such that $m \leq f(x) \leq M$ where $f(x) = e^{-x^2}$, $x \in [0,1]$

$0 \leq x \leq 1$, then

$0 \leq x^2 \leq 1$, then

$-1 \leq -x^2 \leq 0$, then

$e^{-1} \leq e^{-x^2} \leq e^0$ (as $y = e^t$ is an increasing fct)

so $m = e^{-1}$, $M = e^0 = 1$

Therefore,

$$e^{-1}(1-0) \leq \int_0^1 e^{-x^2}\, dx \leq 1(1-0)$$

so $0.367 \leq \int_0^1 e^{-x^2}\, dx \leq 1$