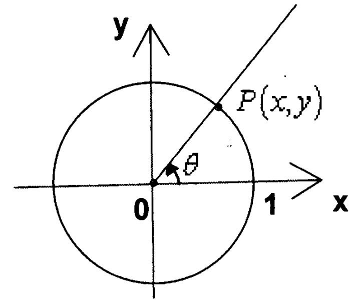
4.1 & 4.2 Graphing Trigonometric Functions

Periods of trigonometric functions

Definition (4.1) A function $y = f(t)$ is **periodic** if there is a positive number p such that

$$f(t+p) = f(t) \text{ for any t.}$$

The smallest number p with the above property is called the **period of the function**.



Recall that the terminal point $P(x, y)$ on the unit circle determined by the central angle θ is the same as the terminal point determined by the central angle $\theta + 2\pi$.

Because sine and cosine functions are defined in terms of the coordinates of $P(x, y)$, their values are unchanged by the addition of any integer multiple of 2π

$$\boxed{\begin{aligned} \sin\theta &= \sin(\theta + 2\pi k), \text{ any } k \in \mathbb{Z} \\ \cos\theta &= \cos(\theta + 2\pi k), \text{ any } k \in \mathbb{Z} \end{aligned}}$$

Property The functions sine and cosine have period 2π. $\boxed{T = 2\pi}$

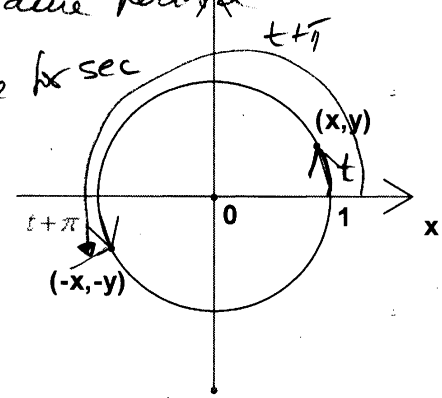
$\sin(\theta + 2\pi) = \sin\theta$

$\cos(\theta + 2\pi) = \cos\theta$

Exercise #1 Find the period of cosecant and secant functions.

$\csc\theta = \dfrac{1}{\sin\theta}$, therefore cosecant has the same period as sine

$\csc(\theta + 2\pi) = \dfrac{1}{\sin(\theta+2\pi)} = \dfrac{1}{\sin\theta} = \csc\theta$ same for sec



Property The functions tangent and cotangent have period π.

$\tan(t + \pi) = \dfrac{-y}{-x} = \dfrac{y}{x} = \tan t$

$\cot(t + \pi) = \dfrac{-x}{-y} = \dfrac{x}{y} = \cot t$

Exercise #2 Find : Use $T = 2\pi$ or $360°$ for sine and cosine

a) $\sin(360° + 23°) = \sin 23° \approx .39$

b) $\cos(45° - 720°) =$
$= \cos(45° - 360°(2))$
$= \cos(45°) = \dfrac{\sqrt{2}}{2}$

c) $\sin\left(\dfrac{19\pi}{3}\right) = \sin\left(6\pi + \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

d) $\tan\left(\dfrac{\pi}{4} + \pi\right) = \tan\dfrac{\pi}{4} = 1$

Use $T = \pi$ for tangent

Trigonometric graphs

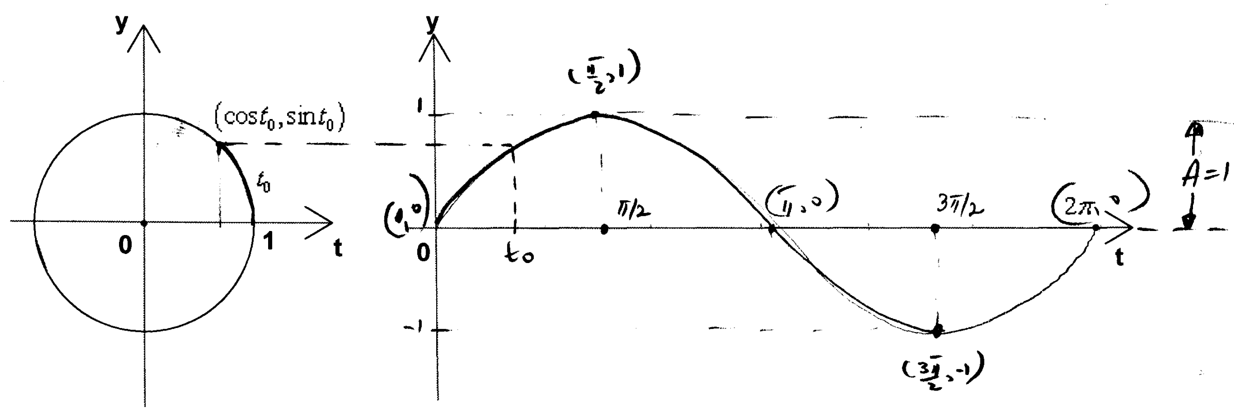
The graph of a function helps us get a better idea of its behavior. We will sketch graphs of the sine and cosine functions and certain transformations of these functions. We will also graph the other trigonometric functions.

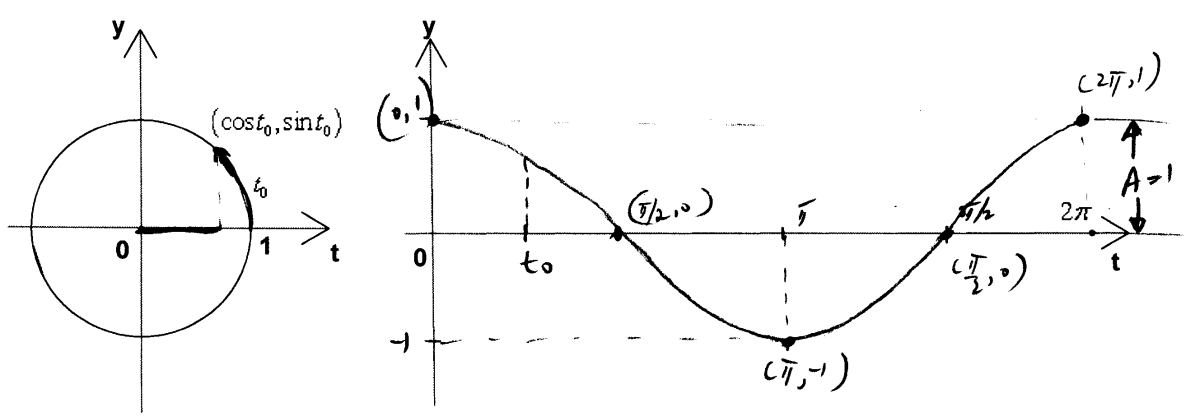
Graphs of the sine and cosine functions

Let t be the central angle measured in radians on the unit circle (t is also equal to the length of the arc).
Sine and cosine functions repeat their values in any interval of length 2π . To sketch their graphs we first sketch the graph of one period, when $0 \le t \le 2\pi$.

The variations of sine and cosine for t between 0 and 2π

t	sin$t$	cos$t$
$0 \to \frac{\pi}{2}$	$0 \to 1$	$1 \to 0$
$\frac{\pi}{2} \to \pi$	$1 \to 0$	$0 \to -1$
$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$-1 \to 0$
$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$0 \to 1$





Definition The **amplitude** of the graph of y is defined as

$$A = \frac{1}{2}|M - m|$$

where M is the greatest value of y and m is the least value of y.

For $y = \sin t$
$y = \cos t$
$A = 1$

Exercise #3 Graph the following functions

a) $y = \sin x$

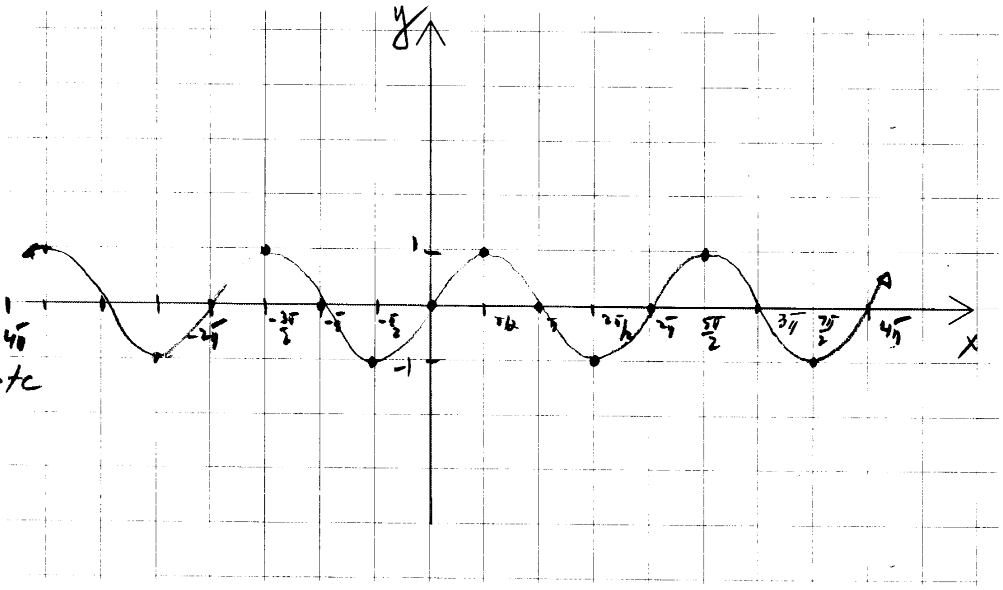
b) $y = \cos x$.

Then state the domain, range, period, amplitude, intercepts, maximum, minimum, and the type of symmetry for each function.

$y = \sin x$

Domain: $x \in \mathbb{R}$
Range: $y \in [-1,1]$ } $\Leftrightarrow$

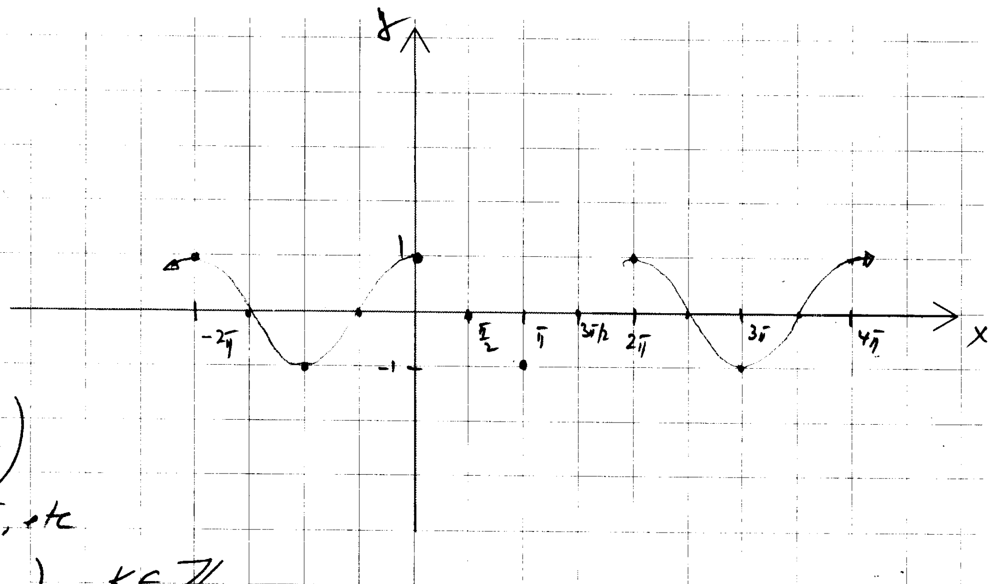
- $\sin : \mathbb{R} \to [-1,1]$
- $T = 2\pi$
- $A = 1$
- max $= 1$ at at $\frac{\pi}{2}, \frac{5\pi}{2}$, etc
  min $= -1$ at at $\frac{3\pi}{2}, \frac{7\pi}{2}$, etc
- sine = odd fct (symmetry about origin)
- x-n: at $0, \pi, 2\pi, 3\pi, \ldots$
  x-n: $(k\pi, 0)$, where $k \in \mathbb{Z}$
- y-n: $(0,0)$



$y = \cos x$

Domain: $x \in \mathbb{R}$
Range: $y \in [-1,1]$ } $\Leftrightarrow$

- $\cos : \mathbb{R} \to [-1,1]$
- $T = 2\pi$
- $A = 1$
- max $= 1$ at $x = 2\pi k$
  min $= -1$ at $x = (2k+1)\pi$
- cosine = even function (symmetry about y-axis)
- x-n: at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc
  x-n: $\left(\frac{(2k+1)\pi}{2}, 0\right)$, $k \in \mathbb{Z}$
- y-n: $(0,1)$



Graphs of transformations of sine and cosine (4.2)

Vertical translations

Exercise #4 a) Graph $y = 2 + \cos x$ over one period. State the domain, range, period, amplitude, intercepts, max, min.
b) Graph $y = \sin x - 1$ over one period. State the domain, range, period, amplitude, intercepts, max, min.

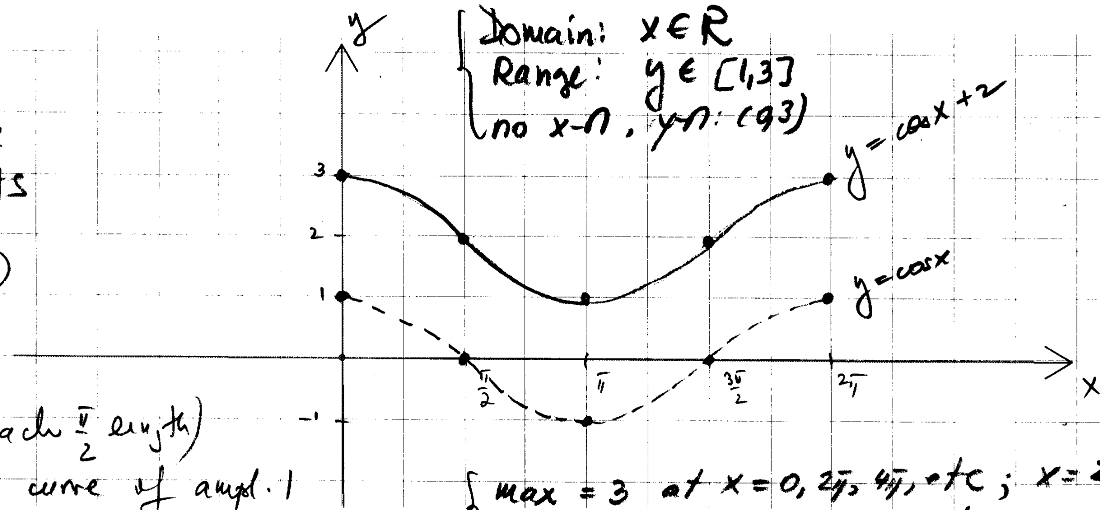
(a) $y = 2 + \cos x$

1st $y = \cos x$
2nd $y = \cos x + 2$
shift up 2 units

$T = 2\pi$, $A = 1$

- Take $[0, 2\pi]$,
- divide it into 4 equal parts (each $\frac{\pi}{2}$ length)
- sketch a cosine curve of ampl. 1
- shift up 2 units

Domain: $x \in \mathbb{R}$
Range: $y \in [1, 3]$
no x-∩, y-∩: (0, 3)



max = 3 at $x = 0, 2\pi, 4\pi$, etc ; $x = 2\pi k$
min = 1 at $x = \pi, 3\pi, 5\pi$, etc ; $x = (2k+1)\pi$

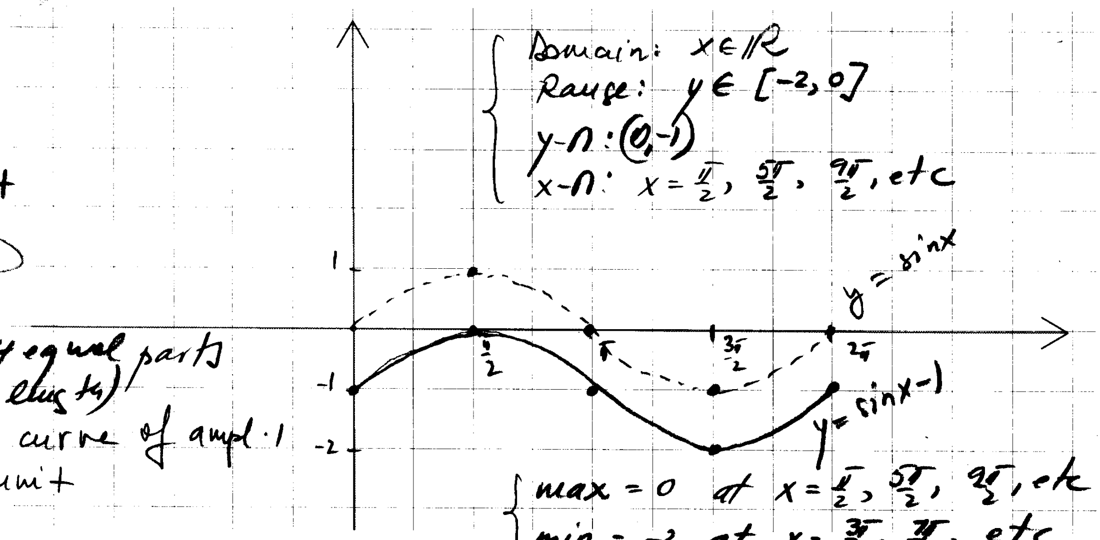
(b) $y = \sin x - 1$

1st $y = \sin x$
2nd $y = \sin x - 1$
shift down 1 unit

$T = 2\pi$, $A = 1$

- take $[0, 2\pi]$
- divide it into 4 equal parts (each of $\frac{\pi}{2}$ length)
- sketch a sine curve of ampl. 1
- shift down 1 unit

Domain: $x \in \mathbb{R}$
Range: $y \in [-2, 0]$
y-∩: (0, -1)
x-∩: $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$, etc



max = 0 at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$, etc
min = -2 at $x = \frac{3\pi}{2}, \frac{7\pi}{2}$, etc

VERTICAL SHIFTING : A vertical shifting does not change the shape of the graph but simply translates it to another position in the plane. Period and amplitude do not change.

Equation	How to obtain the graph	Example
$y = f(x) + k$ $k > 0$	Shift graph of $y = f(x)$ upward k units.	$y = \cos x + 2$
$y = f(x) - k$ $k > 0$	Shift graph of $y = f(x)$ downward k units.	$y = \sin x - 1$

Vertical stretching and compression

Exercise #5 Graph each function and state the domain, range, period, amplitude, intercepts, max, min.

a) $y = 2\sin x$

b) $y = \frac{1}{2}\sin x$

(a) $y = 2\sin x$

1st $y = \sin x$

2nd $y = 2\sin x$

vertical stretch

$T = 2\pi$, $A = 2$

- take $[0, 2\pi]$
- divide it into 4 equal intervals (each of length $\frac{\pi}{2}$)
- sketch a sine curve of amplitude 2

Domain: $x \in \mathbb{R}$

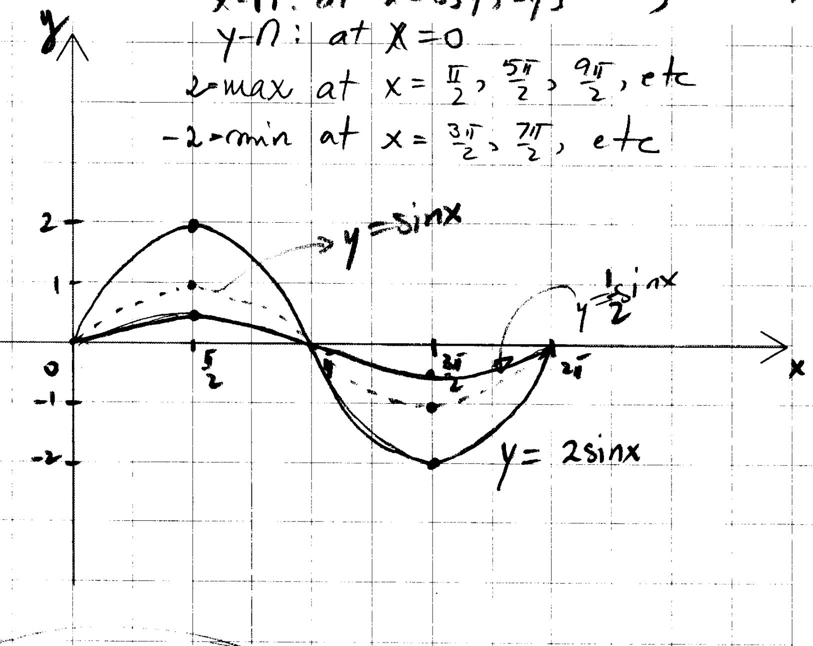
Range: $y \in [-2, 2]$

x-∩: at $x = 0, \pi, 2\pi$, etc; $x = k\pi$

y-∩: at $x = 0$

2 - max at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$, etc

−2 - min at $x = \frac{3\pi}{2}, \frac{7\pi}{2}$, etc



(b) $y = \frac{1}{2}\sin x$ $T = 2\pi$, $A = \frac{1}{2}$

- take $[0, 2\pi]$
- divide it into 4 equal intervals (each of length $\frac{\pi}{2}$)
- sketch a sine curve of amplitude $\frac{1}{2}$

Domain: $x \in \mathbb{R}$

Range: $y \in [-\frac{1}{2}, \frac{1}{2}]$

VERTICAL STRETCHING AND COMPRESSION - period does not change; amplitude changes.

Equation	How to obtain the graph	Example
$y = af(x)$ $a > 1$	Stretch the graph of $y = f(x)$ vertically by a factor of a.	$y = 2\sin x$
$y = af(x)$ $0 < a < 1$	Compress the graph of $y = f(x)$ vertically by a factor of a.	$y = \frac{1}{2}\sin x$

Reflection about the x-axis

Exercise #6 Graph each function over one period and state the domain, range, period, amplitude, intercepts, max, min.

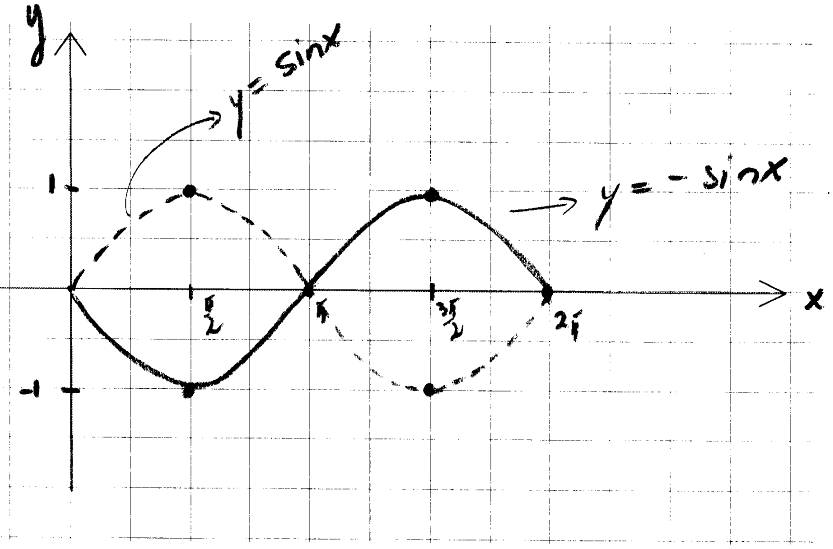
a) $y = -\sin x$

b) $y = -3\cos x$

(a) $y = -\sin x$

$T = 2\pi$, $A = 1$

- take $[0, 2\pi]$
- divide it into 4 equal parts (each of length $\frac{\pi}{2}$)
- sketch a sine curve of amplitude 1
- reflect the graph about x-axis



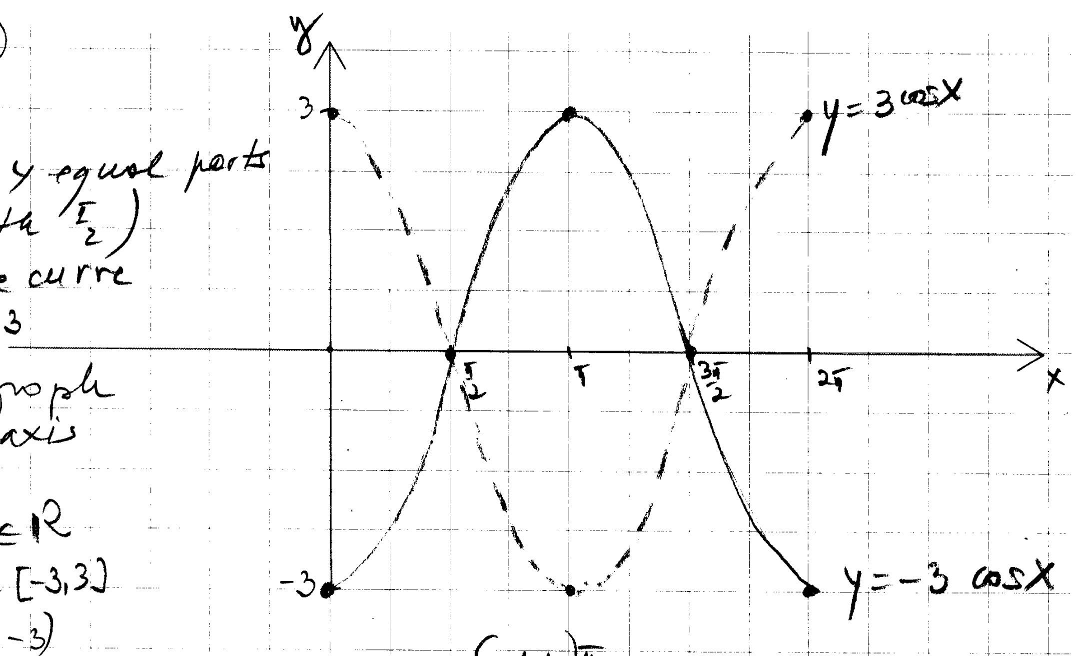
Domain
Range
x-n, y-n
period, ampl

} the same as $y = \sin x$

(b) $y = -3\cos x$

$T = 2\pi$, $A = 3$

- take $[0, 2\pi]$
- divide it into 4 equal parts (each of length $\frac{\pi}{2}$)
- sketch a cosine curve of amplitude 3
- reflect the graph about the x-axis



Domain: $x \in \mathbb{R}$

Range: $y \in [-3, 3]$

y-n: $(0, -3)$

x-n: at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc; $x = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$

Horizontal stretching and compression

Exercise #7 Graph each function over one period and state the domain, range, period, amplitude, intercepts, max, min.

a) $y=\sin(2x)$

b) $y=\sin\left(\frac{1}{2}x\right)$

(a) $y=\sin(2x)$

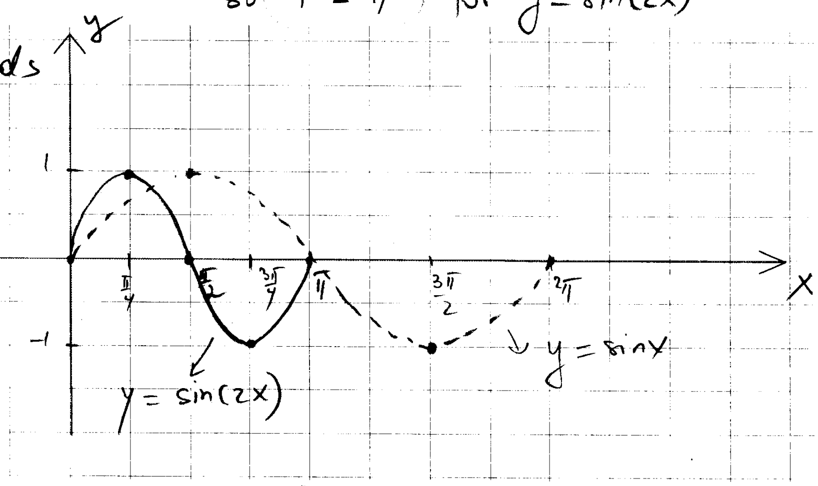
Note that $y=\sin x$ completes one period as $0 \le x \le 2\pi$

while $y=\sin(2x)$ completes one period as $0 \le 2x \le 2\pi$

$0 \le x \le \pi$

so $T=\pi$ for $y=\sin(2x)$

- take $[0,\pi]$
- divide it into 4 equal intervals (each of length $\frac{\pi}{4}$)
- sketch a sine curve of amplitude 1



Domain: $x \in \mathbb{R}$

Range: $y \in [-1,1]$

$T=\pi$

$A=1$

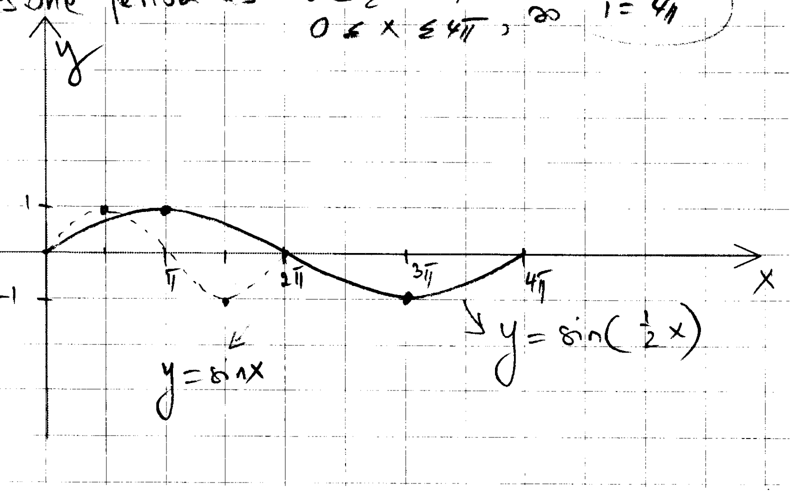
(b) $y=\sin\left(\frac{1}{2}x\right)$

$y=\sin x$ completes one period as $0 \le x \le 2\pi$

$y=\sin\left(\frac{1}{2}x\right)$ completes one period as $0 \le \frac{1}{2}x \le 2\pi$

$0 \le x \le 4\pi$, so $T=4\pi$

- take $[0,4\pi]$
- divide it into 4 equal intervals (each of length π)
- sketch a sine curve of amplitude 1



Domain: $x \in \mathbb{R}$

Range: $y \in [-1,1]$

$T=4\pi$

$A=1$

HORIZONTAL STRETCHING AND COMPRESSION – period changes; amplitude does not change.

Equation	How to obtain the graph	Example
$y=f(ax)$ $a>1$	Compress the graph of $y=f(x)$ horizontally by a factor of $\frac{1}{a}$.	$y=\sin(2x)$
$y=f(ax)$ $0<a<1$	Stretch the graph of $y=f(x)$ horizontally by a factor of $\frac{1}{a}$.	$y=\sin\left(\frac{1}{2}x\right)$

In general, if $y = a\sin(\omega x)$, $y = a\cos(\omega x)$ the amplitude is $A = |a|$ and the period is $T = \frac{2\pi}{\omega}$.

<u>Exercise #8</u> Find the amplitude and the period and sketch a graph for each function over one period,

a) $y = \cos 2x$

b) $y = 4\cos 3x$

c) $y = -2\sin\frac{1}{2}x$

d) $y = \frac{3}{2}\sin\left(-\frac{2}{3}x\right)$

e) $y = -2\cos\left(\frac{\pi}{2}x\right)$

f) $y = 4 + 4\sin 2x$

Graphs of tangent, cotangent, secant and cosecant

Periodic properties The functions tangent and cotangent have period π:

$$\tan(\theta+\pi)=\tan\theta$$

$$\cot(\theta+\pi)=\cot\theta$$

The functions cosecant and secant have period 2π:

$$\sec(\theta+2\pi)=\sec\theta$$

$$\csc(\theta+2\pi)=\csc\theta$$

The graph of the tangent function $\boxed{y=\tan x}=\frac{\sin x}{\cos x}$

What is the domain of the tangent function?

Condition: $\cos x \neq 0$

$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$, etc

$x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$

$x \in \mathbb{R} \setminus \left\{ \frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z} \right\}$

What are the asymptotes of the graph?

V.A. $x=\frac{(2k+1)\pi}{2}, \quad k \in \mathbb{Z}$

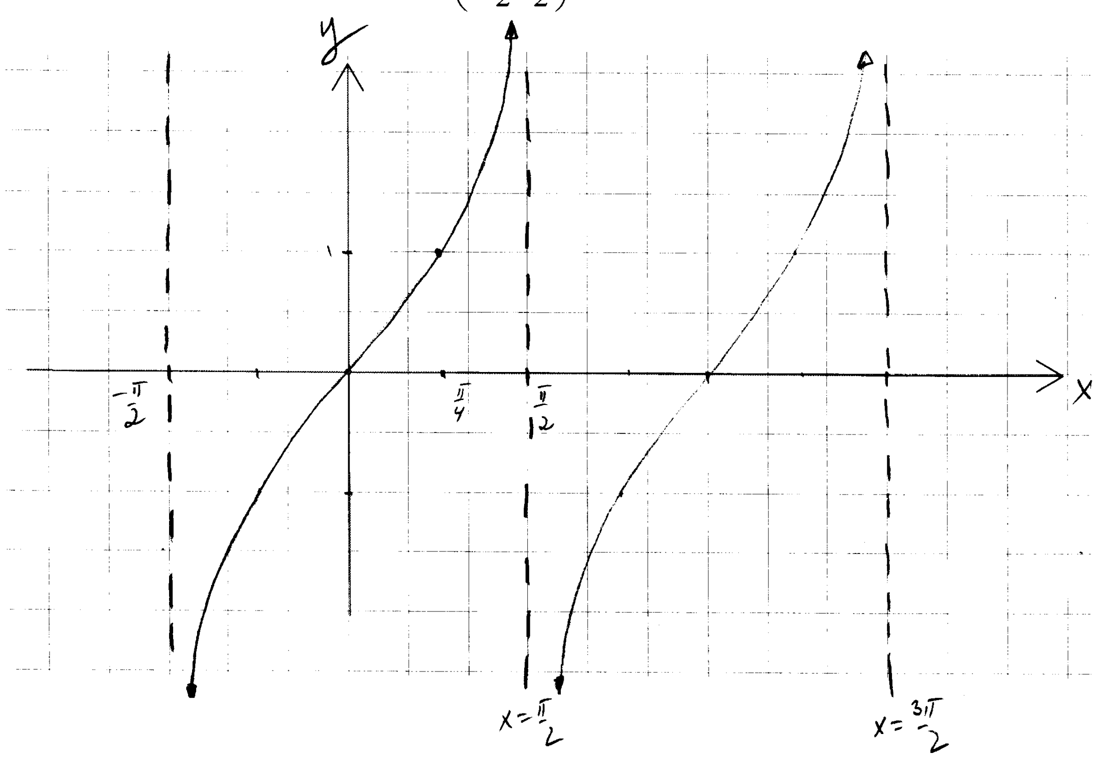
What kind of symmetry does the graph of the tangent function have?

tangent = odd function, so the graph has symmetry about the origin

x	$\tan x$
0	0
$\frac{\pi}{6}$	0.58
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	1.73
$\frac{\pi}{2}$	$y \to \infty$

Since it has a period of π, we need only to sketch the graph on any interval of length π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Range: $y \in \mathbb{R}$



The graph of the cotangent function $\boxed{y = \cot x} = \frac{\cos x}{\sin x}$

x	cotx
0	$y \to \infty$
$\frac{\pi}{6}$	1.73
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	0.58
$\frac{\pi}{2}$	0

What is the domain of the cotangent function?

Condition: $\sin x \neq 0$
$x \neq 0, \pi, 2\pi$, etc
$x \neq k\pi,\ k \in \mathbb{Z}$

$x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$

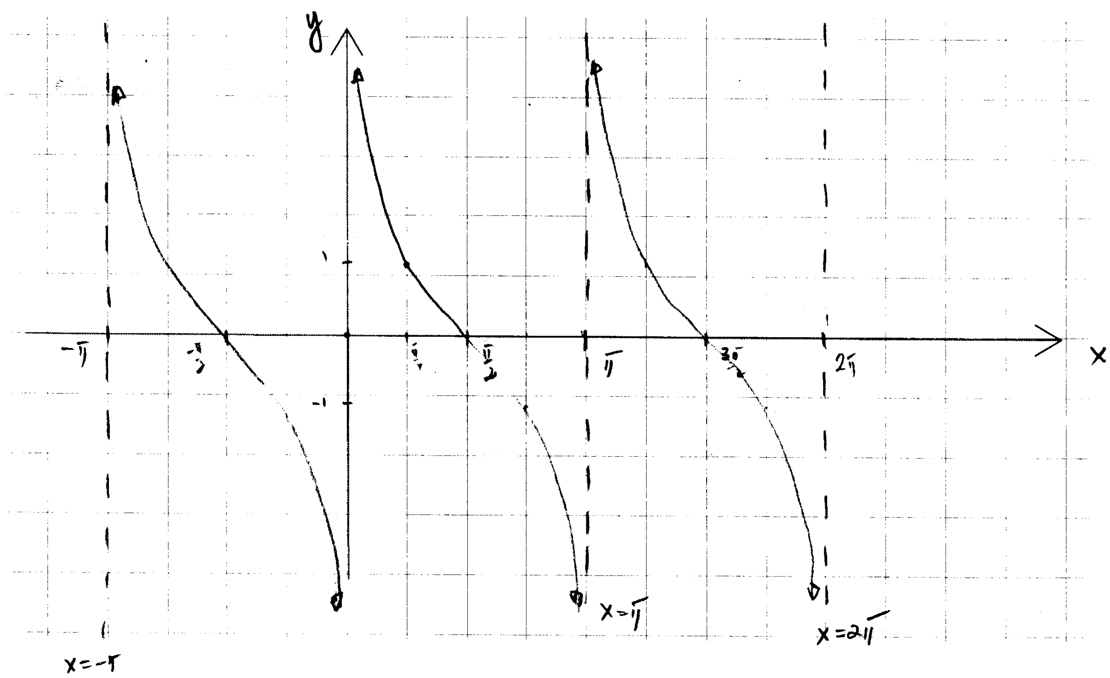
What are the asymptotes of the graph?

V. A. $x = k\pi,\ k \in \mathbb{Z}$

What kind of symmetry does the graph of the cotangent function have?

cotangent = odd function, so the graph has symmetry about the origin

Since it has a period of π, we need only to sketch the graph on any interval of length π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $(0, \pi)$.



What is the range of the function?

$y \in \mathbb{R}$

What are the intercepts?

x-in: at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, etc . $\left(\frac{(2k+1)\pi}{2}, 0\right)$, $k \in \mathbb{Z}$

y-in: none

The graph of the cosecant function

$\boxed{y = \csc x} = \frac{1}{\sin x}$

What is the domain?

Condition: $\sin x \neq 0$

$x \neq k\pi,\ k \in \mathbb{Z}$

$x \in \mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$

What are the asymptotes of the graph?

V. A. $x = k\pi,\ k \in \mathbb{Z}$

What kind of symmetry does the graph of the cosecant function have?

cosecant = odd function, therefore symmetry about the origin

What is the range?

$\csc x = \frac{1}{\sin x}$

$|\sin x| \leq 1$

$\frac{1}{|\sin x|} \geq 1$

$|\csc x| \geq 1$

$\csc x \leq -1$ OR $\csc x \geq 1$ $\quad y \in (-\infty, -1] \cup [1, \infty)$

The graph of the secant function

$\boxed{y = \sec x} = \frac{1}{\cos x}$

What is the domain?

Condition: $\cos x \neq 0$

$x \neq \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$

$x \in \mathbb{R} \setminus \{\frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z}\}$

What are the asymptotes of the graph?

V. A. $x = \frac{(2k+1)\pi}{2},\ k \in \mathbb{Z}$

What kind of symmetry does the graph of the secant function have?

secant = even function, so symmetry about the y-axis

What is the range?

$\sec x = \frac{1}{\cos x}$

$|\cos x| \leq 1$

$\frac{1}{|\cos x|} \geq 1$

$|\sec x| \geq 1$

$\sec x \leq -1$ OR $\sec x \geq 1$

$y \in (-\infty, -1] \cup [1, \infty)$

Since it has a period of 2π, we need only to sketch the graph on any interval of length 2π and then repeat the pattern to the left and to the right. We sketch the graph on the interval $(0, 2\pi)$.