6.4 Special Factoring Rules

OBJECTIVES

1. Factor a difference of squares.
2. Factor a perfect square trinomial.
3. Factor a difference of cubes.
4. Factor a sum of cubes.

By reversing the rules for multiplication of binomials from the last chapter, we get rules for factoring polynomials in certain forms.

OBJECTIVE 1 Factor a difference of squares. The formula for the product of the sum and difference of the same two terms is

$$(x + y)(x - y) = x^2 - y^2.$$

Reversing this rule leads to the following special factoring rule.

Factoring a Difference of Squares

$$x^2 - y^2 = (x + y)(x - y)$$

For example,

$$m^2 - 16 = m^2 - 4^2 = (m + 4)(m - 4).$$

As the next examples show, the following conditions must be true for a binomial to be a difference of squares.

1. Both terms of the binomial must be squares, such as
$$x^2, \qquad 9y^2, \qquad 25, \qquad 1, \qquad m^4.$$
2. The second terms of the binomials must have different signs (one positive and one negative).

EXAMPLE 1 Factoring Differences of Squares

Factor each binomial, if possible.

$$x^2 - y^2 = (x + y)(x - y)$$

(a) $a^2 - 49 = a^2 - 7^2 = (a + 7)(a - 7)$

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(b) $y^2 - m^2 = (y + m)(y - m)$

(c) $z^2 - \frac{9}{16} = z^2 - \left(\frac{3}{4}\right)^2 = \left(z + \frac{3}{4}\right)\left(z - \frac{3}{4}\right)$

(d) $x^2 - 8$

Because 8 is not the square of an integer, this binomial is not a difference of squares. It is a prime polynomial.

(e) $p^2 + 16$

Since $p^2 + 16$ is a *sum* of squares, it is not equal to $(p + 4)(p - 4)$. Also, using FOIL,

$$(p - 4)(p - 4) = p^2 - 8p + 16 \neq p^2 + 16$$

and

$$(p + 4)(p + 4) = p^2 + 8p + 16 \neq p^2 + 16,$$

so $p^2 + 16$ is a prime polynomial.

Now Try Exercises 7, 9, and 11.

CAUTION As Example 1(e) suggests, after any common factor is removed, a *sum* of squares cannot be factored.

EXAMPLE 2 Factoring Differences of Squares

Factor each difference of squares.

$$x^2 \quad - \quad y^2 = (x + y)(x - y)$$

(a) $25m^2 - 16 = (5m)^2 - 4^2 = (5m + 4)(5m - 4)$

(b) $49z^2 - 64 = (7z)^2 - 8^2 = (7z + 8)(7z - 8)$

Now Try Exercise 13.

NOTE As in previous sections, you should always check a factored form by multiplying.

EXAMPLE 3 Factoring More Complex Differences of Squares

Factor completely.

(a) $81y^2 - 36$

First factor out the common factor, 9.

$$\begin{aligned} 81y^2 - 36 &= 9(9y^2 - 4) && \text{Factor out 9.} \\ &= 9[(3y)^2 - 2^2] \\ &= 9(3y + 2)(3y - 2) && \text{Difference of squares} \end{aligned}$$

(b) $9x^2 - 4z^2 = (3x)^2 - (2z)^2 = (3x + 2z)(3x - 2z)$

(c) $p^4 - 36 = (p^2)^2 - 6^2 = (p^2 + 6)(p^2 - 6)$

Neither $p^2 + 6$ nor $p^2 - 6$ can be factored further.

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(d) $m^4 - 16 = (m^2)^2 - 4^2$

$= (m^2 + 4)(m^2 - 4)$ Difference of squares

$= (m^2 + 4)(m + 2)(m - 2)$ Difference of squares again

Now Try Exercises 17, 21, and 25.

CAUTION Remember to factor again when any of the factors is a difference of squares, as in Example 3(d). Check by multiplying.

OBJECTIVE 2 Factor a perfect square trinomial. The expressions 144, $4x^2$, and $81m^6$ are called *perfect squares* because

$$144 = 12^2, \qquad 4x^2 = (2x)^2, \qquad \text{and} \qquad 81m^6 = (9m^3)^2.$$

A **perfect square trinomial** is a trinomial that is the square of a binomial. For example, $x^2 + 8x + 16$ is a perfect square trinomial because it is the square of the binomial $x + 4$:

$$x^2 + 8x + 16 = (x + 4)(x + 4) = (x + 4)^2.$$

For a trinomial to be a perfect square, *two of its terms must be perfect squares.* For this reason, $16x^2 + 4x + 15$ is not a perfect square trinomial because only the term $16x^2$ is a perfect square.

On the other hand, even if two of the terms are perfect squares, the trinomial may not be a perfect square trinomial. For example, $x^2 + 6x + 36$ has two perfect square terms, x^2 and 36, but it is not a perfect square trinomial. (Try to find a binomial that can be squared to give $x^2 + 6x + 36$.)

We can multiply to see that the square of a binomial gives one of the following perfect square trinomials.

Factoring Perfect Square Trinomials

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

The middle term of a perfect square trinomial is always twice the product of the two terms in the squared binomial (as shown in Section 5.6). Use this rule to check any attempt to factor a trinomial that appears to be a perfect square.

EXAMPLE 4 Factoring a Perfect Square Trinomial

Factor $x^2 + 10x + 25$.

The term x^2 is a perfect square, and so is 25. Try to factor the trinomial as

$$x^2 + 10x + 25 = (x + 5)^2.$$

To check, take twice the product of the two terms in the squared binomial.

$$2 \cdot x \cdot 5 = 10x$$

Twice

First term of binomial

Last term of binomial

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Since $10x$ is the middle term of the trinomial, the trinomial is a perfect square and can be factored as $(x + 5)^2$. Thus,

$$x^2 + 10x + 25 = (x + 5)^2.$$

Now Try Exercise 33.

EXAMPLE 5 Factoring Perfect Square Trinomials

Factor each trinomial.

(a) $x^2 - 22x + 121$

The first and last terms are perfect squares ($121 = 11^2$ or $(-11)^2$). Check to see whether the middle term of $x^2 - 22x + 121$ is twice the product of the first and last terms of the binomial $x - 11$.

$$\underset{\substack{\uparrow\\ \text{Twice}}}{2} \cdot \underset{\substack{\uparrow\\ \text{First}\\ \text{term}}}{x} \cdot (\underset{\substack{\uparrow\\ \text{Last}\\ \text{term}}}{-11}) = -22x$$

Since twice the product of the first and last terms of the binomial is the middle term, $x^2 - 22x + 121$ is a perfect square trinomial and

$$x^2 - 22x + 121 = (x - 11)^2.$$

Notice that the sign of the second term in the squared binomial is the same as the sign of the middle term in the trinomial.

(b) $9m^2 - 24m + 16 = (3m)^2 + \underset{\substack{\uparrow\\ \text{Twice}}}{2}(\underset{\substack{\uparrow\\ \text{First}\\ \text{term}}}{3m})(\underset{\substack{\uparrow\\ \text{Last}\\ \text{term}}}{-4}) + (-4)^2 = (3m - 4)^2$

(c) $25y^2 + 20y + 16$

The first and last terms are perfect squares.

$$25y^2 = (5y)^2 \qquad \text{and} \qquad 16 = 4^2$$

Twice the product of the first and last terms of the binomial $5y + 4$ is

$$2 \cdot 5y \cdot 4 = 40y,$$

which is not the middle term of $25y^2 + 20y + 16$. This trinomial is not a perfect square. In fact, the trinomial cannot be factored even with the methods of the previous sections; it is a prime polynomial.

(d) $12z^3 + 60z^2 + 75z$

Factor out the common factor, $3z$, first.

$$\begin{aligned} 12z^3 + 60z^2 + 75z &= 3z(4z^2 + 20z + 25) \\ &= 3z[(2z)^2 + 2(2z)(5) + 5^2] \\ &= 3z(2z + 5)^2 \end{aligned}$$

Now Try Exercises 35, 43, and 51.

NOTE As mentioned in Example 5(a), the sign of the second term in the squared binomial is always the same as the sign of the middle term in the

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trinomial. Also, the first and last terms of a perfect square trinomial must be *positive*, because they are squares. For example, the polynomial $x^2 - 2x - 1$ cannot be a perfect square because the last term is negative.

Perfect square trinomials can also be factored using grouping or FOIL, although using the method of this section is often easier.

OBJECTIVE 3 Factor a difference of cubes. Just as we factored the difference of squares in Objective 1, we can also factor the **difference of cubes** using the following pattern.

Factoring a Difference of Cubes

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

This pattern *should be memorized.* Multiply on the right to see that the pattern gives the correct factors.

$$\begin{array}{rrrr} & x^2 & + xy & + y^2 \\ & & x & - y \\ \hline & - x^2y & - xy^2 & - y^3 \\ x^3 & + x^2y & + xy^2 & \\ \hline x^3 & & & - y^3 \end{array}$$

Notice the pattern of the terms in the factored form of $x^3 - y^3$.

- $x^3 - y^3 =$ (a binomial factor)(a trinomial factor)
- The binomial factor has the difference of the cube roots of the given terms.
- The terms in the trinomial factor are all positive.
- What you write in the binomial factor determines the trinomial factor:

$$x^3 - y^3 = (x - y)\Big(\overset{\text{First term squared}}{x^2} \overset{+}{+} \overset{\text{positive product of the terms}}{xy} \overset{+}{+} \overset{\text{second term squared}}{y^2}\Big).$$

CAUTION The polynomial $x^3 - y^3$ is not equivalent to $(x - y)^3$, because $(x - y)^3$ can also be written as

$$\begin{aligned} (x - y)^3 &= (x - y)(x - y)(x - y) \\ &= (x - y)(x^2 - 2xy + y^2) \end{aligned}$$

but

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

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YOU TRY IT

VIDEO

EXAMPLE 6 Factoring Differences of Cubes

Factor the following.

(a) $m^3 - 125$

Let $x = m$ and $y = 5$ in the pattern for the difference of cubes.

$$\begin{array}{c} x^3 - y^3 = (x - y)(x^2 + xy + y^2) \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ m^3 - 125 = m^3 - 5^3 = (m - 5)(m^2 + 5m + 5^2) \end{array} \qquad \text{Let } x = m,\ y = 5.$$

$$= (m - 5)(m^2 + 5m + 25)$$

(b) $8p^3 - 27$

Since $8p^3 = (2p)^3$ and $27 = 3^3$,

$$\begin{aligned} 8p^3 - 27 &= (2p)^3 - 3^3 \\ &= (2p - 3)[(2p)^2 + (2p)3 + 3^2] \\ &= (2p - 3)(4p^2 + 6p + 9). \end{aligned}$$

(c) $$\begin{aligned} 4m^3 - 32 &= 4(m^3 - 8) \\ &= 4(m^3 - 2^3) \\ &= 4(m - 2)(m^2 + 2m + 4) \end{aligned}$$

(d) $$\begin{aligned} 125t^3 - 216s^6 &= (5t)^3 - (6s^2)^3 \\ &= (5t - 6s^2)[(5t)^2 + 5t(6s^2) + (6s^2)^2] \\ &= (5t - 6s^2)(25t^2 + 30ts^2 + 36s^4) \end{aligned}$$

Now Try Exercises 59, 63, and 69.

CAUTION A common error in factoring a difference of cubes, such as $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, is to try to factor $x^2 + xy + y^2$. It is easy to confuse this factor with a perfect square trinomial, $x^2 + 2xy + y^2$. Because there is no 2 in $x^2 + xy + y^2$, it is very unusual to be able to further factor an expression of the form $x^2 + xy + y^2$.

OBJECTIVE 4 Factor a sum of cubes. A sum of squares, such as $m^2 + 25$, cannot be factored using real numbers, but a **sum of cubes** can be factored by the following pattern, *which should be memorized.*

AUDIO

Factoring a Sum of Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Compare the pattern for the *sum* of cubes with the pattern for the *difference* of cubes. The only difference between them is the positive and negative signs.

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Positive

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2) \qquad \text{Difference of cubes}$$

Same sign Opposite sign

Positive

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) \qquad \text{Sum of cubes}$$

Same sign Opposite sign

Observing these relationships should help you to remember these patterns.

EXAMPLE 7 Factoring Sums of Cubes

Factor.

(a)
$$\begin{aligned} k^3 + 27 &= k^3 + 3^3 \\ &= (k + 3)(k^2 - 3k + 3^2) \\ &= (k + 3)(k^2 - 3k + 9) \end{aligned}$$

(b)
$$\begin{aligned} 8m^3 + 125 &= (2m)^3 + 5^3 \\ &= (2m + 5)[(2m)^2 - 2m(5) + 5^2] \\ &= (2m + 5)(4m^2 - 10m + 25) \end{aligned}$$

(c)
$$\begin{aligned} 1000a^6 + 27b^3 &= (10a^2)^3 + (3b)^3 \\ &= (10a^2 + 3b)[(10a^2)^2 - (10a^2)(3b) + (3b)^2] \\ &= (10a^2 + 3b)(100a^4 - 30a^2b + 9b^2) \end{aligned}$$

Now Try Exercises 61 and 71.

The methods of factoring discussed in this section are summarized here.

Special Factorizations

Difference of squares	$x^2 - y^2 = (x + y)(x - y)$
Perfect square trinomials	$x^2 + 2xy + y^2 = (x + y)^2$
	$x^2 - 2xy + y^2 = (x - y)^2$
Difference of cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Remember the *sum* of *squares* can be factored only if the terms have a common factor.

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