6.1 The Greatest Common Factor; Factoring by Grouping

OBJECTIVES

1. Find the greatest common factor of a list of terms.
2. Factor out the greatest common factor.
3. Factor by grouping.

Recall from Chapter 1 that to **factor** means to write a quantity as a product. That is, factoring is the opposite of multiplying. For example,

Multiplying	*Factoring*
$6 \cdot 2 = 12,$	$12 = 6 \cdot 2.$
Factors (6, 2) → Product (12)	Product (12) → Factors (6, 2)

Other **factored forms** of 12 are

$$-6(-2), \quad 3 \cdot 4, \quad -3(-4), \quad 12 \cdot 1, \quad \text{and} \quad -12(-1).$$

More than two factors may be used, so another factored form of 12 is $2 \cdot 2 \cdot 3$. The positive integer factors of 12 are

$$1, 2, 3, 4, 6, 12.$$

OBJECTIVE 1 Find the greatest common factor of a list of terms. An integer that is a factor of two or more integers is called a **common factor** of those integers. For example, 6 is a common factor of 18 and 24 since 6 is a factor of both 18 and 24. Other common factors of 18 and 24 are 1, 2, and 3. The **greatest common factor (GCF)** of a list of integers is the largest common factor of those integers. Thus, 6 is the greatest common factor of 18 and 24, since it is the largest of their common factors.

NOTE Factors of a number are also divisors of the number. The greatest common factor is actually the same as the greatest common divisor. There are many rules for deciding what numbers divide into a given number. Here are some especially useful divisibility rules for small numbers. It is surprising how many people do not know them.

A Whole Number Divisible by:	*Must Have the Following Property:*
2	Ends in 0, 2, 4, 6, or 8
3	Sum of its digits is divisible by 3.
4	Last two digits form a number divisible by 4
5	Ends in 0 or 5
6	Divisible by both 2 and 3
8	Last three digits form a number divisible by 8
9	Sum of its digits is divisible by 9.
10	Ends in 0

Recall from Chapter 1 that a prime number has only itself and 1 as factors. In Section 1.1 we factored numbers into prime factors. This is the first step in finding the greatest common factor of a list of numbers. We find the greatest common factor (GCF) of a list of numbers as follows.

Finding the Greatest Common Factor (GCF)

Step 1 **Factor.** Write each number in prime factored form.

Step 2 **List common factors.** List each prime number that is a factor of every number in the list. (If a prime does not appear in one of the prime factored forms, it cannot appear in the greatest common factor.)

Step 3 **Choose smallest exponents.** Use as exponents on the common prime factors the *smallest* exponent from the prime factored forms.

Step 4 **Multiply.** Multiply the primes from Step 3. If there are no primes left after Step 3, the greatest common factor is 1.

EXAMPLE 1 Finding the Greatest Common Factor for Numbers

Find the greatest common factor for each list of numbers.

(a) 30, 45

First write each number in prime factored form.

$$30 = 2 \cdot 3 \cdot 5$$
$$45 = 3 \cdot 3 \cdot 5$$

Use each prime the *least* number of times it appears in *all* the factored forms. There is no 2 in the prime factored form of 45, so there will be no 2 in the greatest common factor. The least number of times 3 appears in all the factored forms is 1, and the least number of times 5 appears is also 1. From this, the

$$\text{GCF} = 3^1 \cdot 5^1 = 15.$$

(b) 72, 120, 432

Find the prime factored form of each number.

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$
$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$$
$$432 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$

The least number of times 2 appears in all the factored forms is 3, and the least number of times 3 appears is 1. There is no 5 in the prime factored form of either 72 or 432, so the

$$\text{GCF} = 2^3 \cdot 3^1 = 24.$$

(c) 10, 11, 14

Write the prime factored form of each number.

$$10 = 2 \cdot 5$$
$$11 = 11$$
$$14 = 2 \cdot 7$$

There are no primes common to all three numbers, so the GCF is 1.

Now Try Exercise 1.

The greatest common factor can also be found for a list of variable terms. For example, the terms x^4, x^5, x^6, and x^7 have x^4 as the greatest common factor because each of these terms can be written with x^4 as a factor.

$$x^4 = 1 \cdot x^4, \quad x^5 = x \cdot x^4, \quad x^6 = x^2 \cdot x^4, \quad x^7 = x^3 \cdot x^4$$

NOTE The exponent on a variable in the GCF is the *smallest* exponent that appears in *all* the common factors.

YOU TRY IT

EXAMPLE 2 Finding the Greatest Common Factor for Variable Terms

Find the greatest common factor for each list of terms.

(a) $21m^7, -18m^6, 45m^8, -24m^5$

$$\begin{aligned} 21m^7 &= 3 \cdot 7 \cdot m^7 \\ -18m^6 &= -1 \cdot 2 \cdot 3^2 \cdot m^6 \\ 45m^8 &= 3^2 \cdot 5 \cdot m^8 \\ -24m^5 &= -1 \cdot 2^3 \cdot 3 \cdot m^5 \end{aligned}$$

First, 3 is the greatest common factor of the coefficients 21, -18, 45, and -24. The smallest exponent on m is 5, so the GCF of the terms is $3m^5$.

VIDEO

(b) $x^4y^2, x^7y^5, x^3y^7, y^{15}$

$$\begin{aligned} x^4y^2 &= x^4 \cdot y^2 \\ x^7y^5 &= x^7 \cdot y^5 \\ x^3y^7 &= x^3 \cdot y^7 \\ y^{15} &= y^{15} \end{aligned}$$

There is no x in the last term, y^{15}, so x will not appear in the greatest common factor. There is a y in each term, however, and 2 is the smallest exponent on y. The GCF is y^2.

(c) $-a^2b, -ab^2$

$$\begin{aligned} -a^2b &= -1a^2b = -1 \cdot 1 \cdot a^2b \\ -ab^2 &= -1ab^2 = -1 \cdot 1 \cdot ab^2 \end{aligned}$$

The factors of -1 are -1 and 1. Since $1 > -1$, the GCF is $1ab$ or ab.

Now Try Exercises 9 and 13.

NOTE In a list of negative terms, sometimes a negative common factor is preferable (even though it is not the greatest common factor). In Example 2(c), for instance, we might prefer $-ab$ as the common factor. In factoring exercises, either answer will be acceptable.

OBJECTIVE 2 Factor out the greatest common factor. Writing a polynomial (a sum) in factored form as a product is called **factoring.** For example, the polynomial

$$3m + 12$$

has two terms, $3m$ and 12. The greatest common factor for these two terms is 3. We can write $3m + 12$ so that each term is a product with 3 as one factor.

$$3m + 12 = 3 \cdot m + 3 \cdot 4$$

Now we use the distributive property.

$$3m + 12 = 3 \cdot m + 3 \cdot 4 = 3(m + 4)$$

The factored form of $3m + 12$ is $3(m + 4)$. This process is called **factoring out the greatest common factor.**

CAUTION The polynomial $3m + 12$ is *not* in factored form when written as

$$3 \cdot m + 3 \cdot 4.$$

The *terms* are factored, but the polynomial is not. The factored form of $3m + 12$ is the *product*

$$3(m + 4).$$

EXAMPLE 3 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $5y^2 + 10y = 5y(y) + 5y(2)$ GCF $= 5y$

$= 5y(y + 2)$ Distributive property

To check, multiply out the factored form: $5y(y + 2) = 5y^2 + 10y$, which is the original polynomial.

(b) $20m^5 + 10m^4 + 15m^3$

The GCF for the terms of this polynomial is $5m^3$.

$$20m^5 + 10m^4 + 15m^3 = 5m^3(4m^2) + 5m^3(2m) + 5m^3(3)$$
$$= 5m^3(4m^2 + 2m + 3)$$

Check: $5m^3(4m^2 + 2m + 3) = 20m^5 + 10m^4 + 15m^3$, which is the original polynomial.

(c) $x^5 + x^3 = x^3(x^2) + x^3(1) = x^3(x^2 + 1)$ Don't forget the 1.

(d) $20m^7p^2 - 36m^3p^4 = 4m^3p^2(5m^4) - 4m^3p^2(9p^2)$ GCF $= 4m^3p^2$

$= 4m^3p^2(5m^4 - 9p^2)$

(e) $\frac{1}{6}n^2 + \frac{5}{6}n = \frac{1}{6}n(n) + \frac{1}{6}n(5) = \frac{1}{6}n(n + 5)$ GCF $= \frac{1}{6}n$

Now Try Exercises 37, 41, and 47.

CAUTION Be sure to include the 1 in a problem like Example 3(c). *Always* check that the factored form can be multiplied out to give the original polynomial.

EXAMPLE 4 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $a(a + 3) + 4(a + 3)$

The binomial $a + 3$ is the greatest common factor here.

Same

$$a(a + 3) + 4(a + 3) = (a + 3)(a + 4)$$

(b) $x^2(x + 1) - 5(x + 1) = (x + 1)(x^2 - 5)$ Factor out $x + 1$.

Now Try Exercise 55.

OBJECTIVE 3 Factor by grouping. When a polynomial has four terms, common factors can sometimes be used to **factor by grouping.**

EXAMPLE 5 Factoring by Grouping

Factor by grouping.

(a) $2x + 6 + ax + 3a$

The first two terms have a common factor of 2, and the last two terms have a common factor of a.

$$2x + 6 + ax + 3a = 2(x + 3) + a(x + 3)$$

The expression is still not in factored form because it is the *sum* of two terms. Now, however, $x + 3$ is a common factor and can be factored out.

$$\begin{aligned} 2x + 6 + ax + 3a &= 2(x + 3) + a(x + 3) \\ &= (x + 3)(2 + a) \end{aligned}$$

The final result is in factored form because it is a *product.* Note that the goal in factoring by grouping is to get a common factor, $x + 3$ here, so that the last step is possible.

Check: $(x + 3)(2 + a) = 2x + 6 + ax + 3a$, which is the original polynomial.

(b) $2x^2 - 10x + 3xy - 15y = (2x^2 - 10x) + (3xy - 15y)$ Group terms.

$= 2x(x - 5) + 3y(x - 5)$ Factor each group.

$= (x - 5)(2x + 3y)$ Factor out the common factor, $x - 5$.

Check: $(x - 5)(2x + 3y) = 2x^2 + 3xy - 10x - 15y$ FOIL

$= 2x^2 - 10x + 3xy - 15y$ Original polynomial

(c) $t^3 + 2t^2 - 3t - 6 = (t^3 + 2t^2) + (-3t - 6)$ Group terms.

$= t^2(t + 2) - 3(t + 2)$ Factor out -3 so there is a common factor, $t + 2$; $-3(t + 2) = -3t - 6$.

$= (t + 2)(t^2 - 3)$ Factor out $t + 2$.

Check by multiplying.

Now Try Exercises 69, 73, and 77.

CAUTION Use negative signs carefully when grouping, as in Example 5(c). Otherwise, sign errors may result. Always check by multiplying.

Use these steps when factoring four terms by grouping.

Factoring by Grouping

Step 1 **Group terms.** Collect the terms into two groups so that each group has a common factor.

Step 2 **Factor within groups.** Factor out the greatest common factor from each group.

Step 3 **Factor the entire polynomial.** Factor a common binomial factor from the results of Step 2.

Step 4 **If necessary, rearrange terms.** If Step 2 does not result in a common binomial factor, try a different grouping.

EXAMPLE 6 Rearranging Terms Before Factoring by Grouping

Factor by grouping.

(a) $10x^2 - 12y + 15x - 8xy$

Factoring out the common factor of 2 from the first two terms and the common factor of x from the last two terms gives

$$10x^2 - 12y + 15x - 8xy = 2(5x^2 - 6y) + x(15 - 8y).$$

This did not lead to a common factor, so we try rearranging the terms. There is usually more than one way to do this. We try

$$10x^2 - 8xy - 12y + 15x,$$

and group the first two terms and the last two terms as follows.

$$\begin{aligned} 10x^2 - 8xy - 12y + 15x &= 2x(5x - 4y) + 3(-4y + 5x) \\ &= 2x(5x - 4y) + 3(5x - 4y) \\ &= (5x - 4y)(2x + 3) \end{aligned}$$

Check: $(5x - 4y)(2x + 3) = 10x^2 + 15x - 8xy - 12y$ FOIL

$= 10x^2 - 12y + 15x - 8xy$ Original polynomial

(b) $2xy + 3 - 3y - 2x$

We need to rearrange these terms to get two groups that each have a common factor. Trial and error suggests the following grouping.

$2xy + 3 - 3y - 2x = (2xy - 3y) + (-2x + 3)$ Group terms.

$= y(2x - 3) - 1(2x - 3)$ Factor each group.

$= (2x - 3)(y - 1)$ Factor out the common binomial factor.

Since the quantities in parentheses in the second step must be the same, we factored out -1 rather than 1. Check by multiplying.

Now Try Exercise 81.