Math 71 Fall 2011 Name:

TEST 2 @ 135 points

Write in a neat and organized fashion. Write your complete solutions on SEPARATE PAPER. You should use a pencil. For an exercise to be complete there needs to be a detailed solution to the problem. Do not just write down an answer. No proof, no credit given! Clearly label each exercise.

1. Do the following:

a)Simplify $\left(\dfrac{a^{\frac{-2}{5}}b^{\frac{2}{3}}}{a^{-\frac{1}{2}}b}\right)^{-4}$ and write the final answer using only positive exponents

b) Simplify: $2\sqrt{75}+4\sqrt{12}-\left(2\sqrt{2}-\sqrt{3}\right)\left(2\sqrt{2}+\sqrt{3}\right)$

c) Simplify: $\dfrac{-24+\sqrt{-180}}{8}$

d) If $f\left(x\right)=x^2$, find and simplify: $f\left(4-\sqrt{x+1}\right)$.

e) Find and simplify: $2^{-3}+4^{\frac{1}{2}}-\left(\dfrac{2}{3}\right)^{-1}$

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2. Let $f\left(x\right)=\dfrac{x+2}{x+3}$ and $g\left(x\right)=\dfrac{x+1}{x^2+2x-3}$ two functions.

a) What is the domain of *f*?

b) What is the domain of *g*?

c) Find all the values of *a* for which $f\left(a\right)=g\left(a\right)+1$.

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3. If $g\left(x\right)=\dfrac{5}{x+2}+\dfrac{25}{x^2+4x+4}$, find all values of *x* for which $g\left(x\right)=20$.

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4. $f\left(x\right)=x^2-3x-8$ Find the following and simplify:

a) $f\left(1-i\right)$ b) $f\left(2+\sqrt{5}\right)$

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5. Let $f\left(x\right)=\sqrt{x-1}$.

a) What is the domain of this function?

b) Sketch the graph of the function by plotting points. Label the axes and all the points used.

c) What is the range of this function?

d) List the *x*-and *y*-intercepts.

6. Solve the following equations:

a) $x^3 - 4x^2 - x + 4 = 0$

b) $(5x+4)(x-1) = 2$

c) $\sqrt{x+5} - \sqrt{x-3} - 2 = 0$

d) Solve for r: $P = \dfrac{A}{1+r}$

e) $3x^4 - 48x^2 = 0$

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7. The height , h(t), in feet, of a gymnasts t seconds after dismounting the uneven parallel bars is given by $h(t) = -16t^2 + 8t + 8$.

a) What was the initial height? Write the answer using function notation.

b) How long will it take the gymnast to reach the ground? Write the answer using function notation.

c) When will the gymnasts be 8 feet above the ground?

d) What is the maximum height and how long does it take the gymnast to get there?

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8. Do the following division using long division and relate dividend, division, quotient, and remainder.

$$\frac{3x^5 - x^3 + 4x^2 - 12x - 8}{x^2 - 2}$$

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9. Solve the following equations in the set of complex numbers. What kind of solutions does each equation have: real, rational, irrational conjugates, or complex conjugates?

a) $x^4 + 2x^2 = 8$ b) $81\left(x + \dfrac{1}{3}\right)^2 + 1 = 0$ c) $x^2 + 4x = 6$

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10. Solve by **completing the square** in the set of complex numbers. What kind of solutions does this equation have: real, rational, irrational conjugates, or complex conjugates? $9p^2 + 12p - 41 = 0$

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11. Let $y = x^2 - 3x - 10$

a) Graph the function. Label all points and the axes. Clearly show how you find the vertex and the intercepts. Graph the axis of symmetry of the parabola.

b) State the domain and range.

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12. Among all pairs of numbers whose sum is 16, find the pair whose product is maximum. What is the maximum product? Do not just guess an answer. Show a coherent mathematical proof.

Extra Credit

1. (3 points) Factor completely: $a^{32} - b^{32}$.
2. (3 points) Factor completely: $x^{2m} + x^{m+n} + x^{m-n} + 1$.
3. (3 points) Solve : $\sqrt{1 + x\sqrt{x^2 + 24}} = x + 1$
4. (3 points) Find the real number m such that the number $3i^3 - 2mi^2 + (1-m)i + 5$ is a real number.

MATH71

TEST 2 - SOLUTIONS

① (a) $\left(\dfrac{a^{-\frac{2}{5}}b^{\frac{2}{3}}}{a^{-\frac{1}{2}}b}\right)^{-4} = \left(a^{-\frac{2}{5}+\frac{1}{2}}\, b^{\frac{2}{3}-1}\right)^{-4}$

$= \left(a^{\frac{1}{10}}b^{-\frac{1}{3}}\right)^{-4} = \left(a^{\frac{1}{10}}\right)^{-4}\left(b^{-\frac{1}{3}}\right)^{-4}$

$= a^{-\frac{4}{10}}b^{\frac{4}{3}} = \boxed{\dfrac{b^{\frac{4}{3}}}{a^{\frac{2}{5}}}}$

(b) $2\sqrt{75}+4\sqrt{12}-(2\sqrt{2}-\sqrt{3})(2\sqrt{2}+\sqrt{3}) =$

$= 2\sqrt{25\cdot 3}+4\sqrt{4\cdot 3}-\left((2\sqrt{2})^2-(\sqrt{3})^2\right)$

$= 2\cdot 5\sqrt{3}+4\cdot 2\sqrt{3}-(4\cdot 2-3)$

$= 10\sqrt{3}+8\sqrt{3}-5$

$= \boxed{18\sqrt{3}-5}$

(c) $\dfrac{-24+\sqrt{-180}}{8} = \dfrac{-24+\sqrt{(-1)3^2 2^2\cdot 5}}{8}$

$= \dfrac{-24+3\cdot 2\sqrt{5}i}{8} = \dfrac{2(-12+3\sqrt{5}i)}{8}$

$= \boxed{\dfrac{-12+3\sqrt{5}i}{4}}$

(d) $f(x)=x^2$

$f(4-\sqrt{x+1}) = (4-\sqrt{x+1})^2$

$= 4^2-2\cdot 4\sqrt{x+1}+(\sqrt{x+1})^2$

$= 16-8\sqrt{x+1}+x+1$

$= \boxed{17-8\sqrt{x+1}+x}$

(e) $2^{-3}+4^{\frac{1}{2}}-\left(\frac{2}{3}\right)^{-1} =$

$= \dfrac{1}{2^3}+\sqrt{4}-\dfrac{1}{\frac{2}{3}}$

$= \dfrac{1}{8}+{}^{8/}\dfrac{2}{1}-{}^{4/}\dfrac{3}{2} = \dfrac{1+16-12}{8}$

$= \boxed{\dfrac{5}{8}}$

② $f(x)=\dfrac{x+2}{x+3}$

$g(x)=\dfrac{x+1}{x^2+2x-3} = \dfrac{x+1}{(x+3)(x-1)}$

(a) condition: $x+3\neq 0$
$x\neq -3$

Domain: $\boxed{x\in\mathbb{R}\setminus\{-3\}}$

(b) Conditions: $\begin{cases} x+3\neq 0 \\ \text{and} \\ x-1\neq 0\end{cases}$

so $x\neq -3,\ x\neq 1$

Domain: $\boxed{x\in\mathbb{R}\setminus\{-3,1\}}$

(c) $f(a)=g(a)+1$

$^{a-1}\dfrac{a+2}{a+3} = \dfrac{a+1}{(a+3)(a-1)} + {}^{(a+3)(a-1)}1$

$\left[\begin{array}{l} a\neq -3,\ a\neq 1 \\ LCD = (a+3)(a-1)\end{array}\right.$

$(a-1)(a+2) = a+1+(a+3)(a-1)$

$\cancel{a^2}+\cancel{a}-2 = \cancel{a}+1+\cancel{a^2}+2a-3$

$-2 = 2a-2$

$0 = 2a \Rightarrow a=0$

$\boxed{a\in\{0\}}$

③ $g(x)=\dfrac{5}{x+2}+\dfrac{25}{x^2+4x+4}$

$g(x)=20 \Rightarrow$

$^{x+2}\dfrac{5}{x+2}+\dfrac{25}{(x+2)^2} = {}^{(x+2)^2}20$

$\left[\begin{array}{l} x\neq -2 \\ LCD=(x+2)^2\end{array}\right.$

$5(x+2)+25=20(x+2)^2$

$5x+10+25=20(x^2+4x+4)$

$5x+35=20x^2+80x+80$

$20x^2+75x+45=0 \quad |\div 5$

$4x^2+15x+9=0$

Method I – solve by factoring

$(4x+3)(x+3)=0$

$4x+3=0$ OR $x+3=0$

$x=-\frac{3}{4}$ $\qquad x=-3$

OR

Method II – solve using the Quadratic Formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-15\pm\sqrt{15^2-4\cdot4\cdot9}}{2\cdot4}=\frac{-15\pm\sqrt{81}}{8}$$

$$x=\frac{-15\pm9}{8} \quad \frac{-15+9}{8}=-\frac{3}{4} \text{ OR } \frac{-15-9}{8}=-3$$

$\therefore \boxed{x\in\{-3,-\frac{3}{4}\}}$

(4) $f(x)=x^2-3x-8$

(a) $f(1-i)=(1-i)^2-3(1-i)-8$

$=1-2i+i^2-3+3i-8$

$=-10+i-1=\boxed{-11+i}$

(b) $f(2+\sqrt{5})=(2+\sqrt{5})^2-3(2+\sqrt{5})-8$

$=4+4\sqrt{5}+5-6-3\sqrt{5}-8$

$=\boxed{-5+\sqrt{5}}$

(5) $f(x)=\sqrt{x-1}$

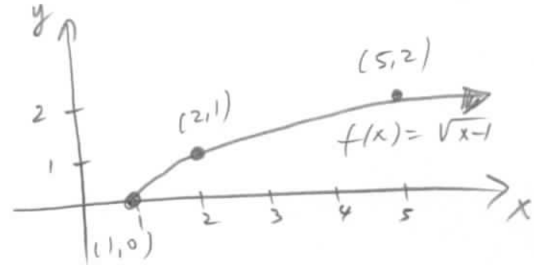
(a) condition: $x-1\geq0$

$\Rightarrow x\geq1$

Domain: $\boxed{x\in[1,\infty)}$

(b)

x	y
1	0
2	1
5	2



(c) Range: $\boxed{y\in[0,\infty)}$

(d) x-in: $(1,0)$

y-in: none

(6) (a) $x^3-4x^2-x+4=0$

3rd-degree equation

we'll solve it by factoring

$x^2(x-4)-(x-4)=0$

$(x-4)(x^2-1)=0$

$(x-4)(x-1)(x+1)=0$

$x-4=0$ OR $x-1=0$ OR $x+1=0$

$x=4 \qquad x=1 \qquad x=-1$

$\therefore \boxed{x\in\{4,1,-1\}}$

(b) $(5x+4)(x-1)=2$

Quadratic equation.
We'll solve it by factoring (if possible) or by the Quadratic formula

$$5x^2-5x+4x-4-2=0$$
$$5x^2-x-6=0$$

I Factoring:
$$(5x-6)(x+1)=0$$
$$5x-6=0 \quad \text{OR} \quad x+1=0$$
$$x=\frac{6}{5} \qquad x=-1$$

or
II Quadratic Formula
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \begin{cases} a=5 \\ b=-1 \\ c=-6 \end{cases}$$

$$x=\frac{1\pm\sqrt{1-4(5)(-6)}}{2\cdot 5}=\frac{1\pm\sqrt{1+120}}{10}$$

$$x=\frac{1\pm 11}{10} \quad \frac{1+11}{10}=\frac{6}{5} \text{ OR } \frac{1-11}{10}=-1$$

so $\boxed{x\in\{\frac{6}{5},-1\}}$

(c) $\sqrt{x+5}-\sqrt{x-3}-2=0$

Radical equation
We'll isolate one radical:

$$\sqrt{x+5}=2+\sqrt{x-3} \quad /^2$$
$$(\sqrt{x+5})^2=(2+\sqrt{x-3})^2$$

$$\cancel{x}+5=4+4\sqrt{x-3}+\cancel{x}-3$$
$$5=1+4\sqrt{x-3}$$

Radical equation.
We'll isolate the radical

$$4\sqrt{x-3}=4$$
$$\sqrt{x-3}=1 \quad /^2$$
$$(\sqrt{x-3})^2=1^2$$
$$x-3=1$$
$$x=4$$

Check $x=4$:
$$\sqrt{4+5}-\sqrt{4-3}-2\overset{?}{=}0$$
$$3-1-2=0 \quad \text{TRUE}$$

Therefore, $\boxed{x\in\{4\}}$

(d) $P=\frac{A}{1+r}$, $r=?$

$$P(1+r)=A$$
$$1+r=\frac{A}{P} \Rightarrow \boxed{r=\frac{A}{P}-1}$$

(e) $3x^4-48x^2=0$

4th degree equation
We'll solve it by factoring.

$$3x^2(x^2-16)=0$$
$$3x^2(x-4)(x+4)=0$$
$$x=0 \quad \text{OR} \quad x-4=0 \quad \text{OR} \quad x+4=0$$
$$x=4 \qquad x=-4$$

so $\boxed{x\in\{0,4,-4\}}$

(7) $h(t) = -16t^2 + 8t + 8$

$\begin{cases} t = \text{time (seconds)} \\ h(t) = \text{height} \end{cases}$

(a) $\boxed{h(0) = 8 \text{ ft}}$ initial height

(b) find t when $h(t) = 0$

solve $-16t^2 + 8t + 8 = 0 \quad | \div(-8)$

$2t^2 - t - 1 = 0$

$(2t+1)(t-1) = 0$

$2t+1 = 0$ OR $t - 1 = 0$

$\cancel{t = -\frac{1}{2}}$ $\quad t = 1$

(not possible, as $t \geq 0$)

$\boxed{h(1) = 0 \text{ ft}}$ it takes the gymnast one second to reach the ground

(c) find t when $h(t) = 8$

solve $-16t^2 + 8t + 8 = 8$

$-16t^2 + 8t = 0$

$8t(-2t+1) = 0$

$t = 0$ OR $-2t + 1 = 0$

(initial time) $\quad t = \frac{1}{2}$ seconds

The gymnast is at 8 ft above the ground initially and, again, after 0.5 seconds.

(d) $h(t) = -16t^2 + 8t + 8$ represents a parabola opening down, so the maximum occurs at the vertex $V(t_v, h_v)$

$t_v = \frac{-b}{2a} = \frac{-8}{2(-16)} = \frac{1}{4} = 0.25$ seconds

$h(\tfrac{1}{4}) = -16(\tfrac{1}{4})^2 + 8(\tfrac{1}{4}) + 8$

$= -16 \cdot \frac{1}{16} + 2 + 8$

$h(\tfrac{1}{4}) = h_{max} = 9 \text{ ft}$

The max. height is 9 ft and it takes 0.25 seconds to get there.

(8)

$$
\begin{array}{r|l}
 & 3x^3 + 5x + 4 \\
\hline
x^2 - 2 & 3x^5 - x^3 + 4x^2 - 12x - 8 \\
 & -3x^5 + 6x^3 \\
\hline
 & 5x^3 + 4x^2 - 12x - 8 \\
 & -5x^3 \qquad\qquad + 10x \\
\hline
 & 4x^2 - 2x - 8 \\
 & -4x^2 \qquad + 8 \\
\hline
 & -2x
\end{array}
$$

Therefore,

$$\boxed{\frac{3x^5 - x^3 + 4x^2 - 12x - 8}{x^2 - 2} = 3x^3 + 5x + 4 - \frac{2x}{x^2 - 2}}$$

(9) (a) $x^4+2x^2=8$

$x^4+2x^2-8=0$

4th-degree equation

We'll use substitution:

let $x^2=t$

then $x^4=t^2$

The equation becomes:

$t^2+2t-8=0$

$(t+4)(t-2)=0$

$t+4=0$ OR $t-2=0$

$t=-4$ $\quad t=2$

if $t=-4$,

$x^2=-4$

$\sqrt{x^2}=\sqrt{-4}$

$x=\pm\sqrt{-4}$

$x=\pm 2i$ complex-conjugate solutions

if $t=2$

$x^2=2$

$\sqrt{x^2}=\sqrt{2}$

$x=\pm\sqrt{2}$ irrational conjugate sols.

$$\boxed{x\in\{\pm 2i, \pm\sqrt{2}\}}$$

(b) $81\left(x+\frac{1}{3}\right)^2+1=0$

Quadratic eq. We'll use the Square Root Property.

$81\left(x+\frac{1}{3}\right)^2=-1$

$\left(x+\frac{1}{3}\right)^2=\frac{-1}{81}$

$\sqrt{\left(x+\frac{1}{3}\right)^2}=\sqrt{\frac{-1}{81}}$

$x+\frac{1}{3}=\pm\sqrt{\frac{-1}{81}}$

$$\boxed{x=-\frac{1}{3}\pm\frac{1}{9}i}$$

complex-conjugate sols.

(c) $x^2+4x=6$

Quadratic eq. We'll use the Quadratic Formula

(Note that x^2+4x-6 is not factorable)

$x^2+4x-6=0$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\quad \begin{cases} a=1 \\ b=4 \\ c=-6 \end{cases}$

$x=\dfrac{-4\pm\sqrt{16-4\cdot 1\cdot(-6)}}{2\cdot 1}$

$x=\dfrac{-4\pm\sqrt{16+24}}{2}=\dfrac{-4\pm\sqrt{40}}{2}$

$x=\dfrac{-4\pm 2\sqrt{10}}{2}=\dfrac{2(-2\pm\sqrt{10})}{2}$

$\boxed{x=-2\pm\sqrt{10}}$ irrational conjugate solutions

(10) $9p^2+12p-41=0$

Make leading coefficient = 1

$$p^2+\frac{12}{9}p-\frac{41}{9}=0$$

$$p^2+\frac{4}{3}p-\frac{41}{9}=0$$

isolate the constant:

$$p^2+\frac{4}{3}p=\frac{41}{9}$$

Find the missing term:

$$\left(\frac{1}{2}\operatorname{coef} p\right)^2=\left(\frac{1}{2}\cdot\frac{4}{3}\right)^2=\left(\frac{2}{3}\right)^2=\frac{4}{9}$$

complete the square and solve:

$$p^2+\frac{4}{3}p+\frac{4}{9}=\frac{41}{9}+\frac{4}{9}$$

$$\left(p+\frac{2}{3}\right)^2=\frac{45}{9}$$

$$\left(p+\frac{2}{3}\right)^2=5$$

$$\sqrt{\left(p+\frac{2}{3}\right)^2}=\sqrt{5}$$

$$p+\frac{2}{3}=\pm\sqrt{5}$$

$$\boxed{p=-\frac{2}{3}\pm\sqrt{5}}$$

irrational conjugates

(11) $y=x^2-3x-10$

(a) parabola opening upward

$V(x_v,y_v)$: $x_v=\frac{-b}{2a}=\frac{3}{2}$

$$y_v=\left(\frac{3}{2}\right)^2-3\left(\frac{3}{2}\right)-10$$

$$y_v=\frac{9}{4}-\frac{9}{2}-10=\frac{9-18-40}{4}$$

$$y_v=\frac{-49}{4}$$

$$\boxed{V\left(\frac{3}{2},\frac{-49}{4}\right)}$$

y-n: $x=0, y=-10$

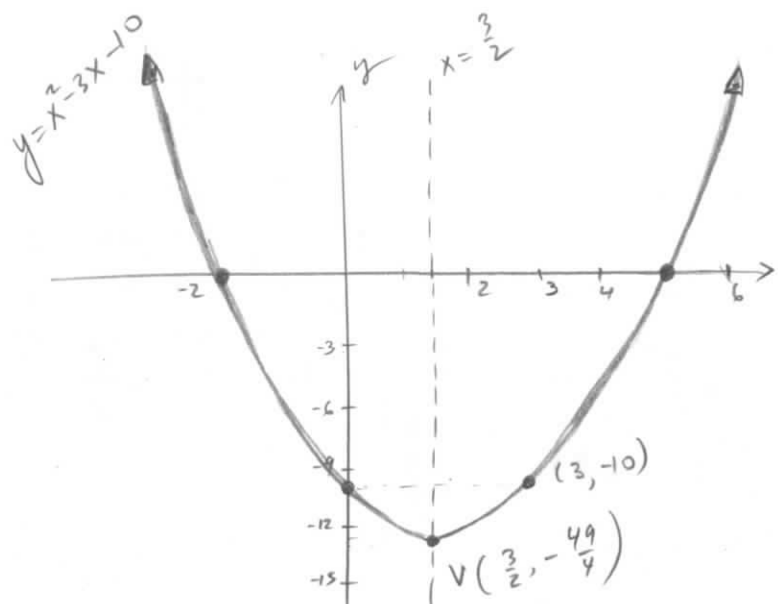
$$\boxed{\text{y-n: } (0,-10)}$$

x-n: $y=0$, $x^2-3x-10=0$

$(x-5)(x+2)=0$

$x-5=0$ OR $x+2=0$

$$\boxed{\text{x-n: } (5,0) \text{ and } (-2,0)}$$



(b) Domain: $x\in\mathbb{R}$

Range: $y\geq-\frac{49}{4}$

(12) Given

two numbers:
their sum = 16
product = maximum

Find

The numbers

Solutions

let x = one number
y = the other number

then $x + y = 16$ (1)

we want xy = maximum

let $P = xy$ (2)

(1) $\Rightarrow$ $y = 16 - x$ (3)

substitute (3) into (2):

$P = x(16-x)$

$P = -x^2 + 16x$

this eq. gives the product in terms of one number, x

The eq. represents a parabola opening down, so the maximum occurs at the vertex

$V(x_v, P_v)$

$$x_v = \frac{-b}{2a} = \frac{-16}{2(-1)} = 8$$

then $P_v = P_{max} = 8(16-8) = 64$

The numbers whose sum is 16 with the maximum product are $\boxed{8 \text{ and } 8}$

Extra credit

(1) $a^{32} - b^{32} = (a^{16})^2 - (b^{16})^2$

$= (a^{16} - b^{16})(a^{16} + b^{16})$

$= ((a^8)^2 - (b^8)^2)(a^{16} + b^{16})$

$= (a^8 - b^8)(a^8 + b^8)(a^{16} + b^{16})$

$= ((a^4)^2 - (b^4)^2)(a^8 + b^8)(a^{16} + b^{16})$

$= (a^4 - b^4)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})$

$= ((a^2)^2 - (b^2)^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})$

$= (a^2 - b^2)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})$

$= \boxed{(a-b)(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})}$

(2) $x^{2m} + x^{m+n} + x^{m-n} + 1 =$

$= x^{m-n}(x^{m+n} + 1) + (x^{m+n} + 1)$

$= \boxed{(x^{m+n} + 1)(x^{m-n} + 1)}$

(3) $\sqrt{1+x\sqrt{x^2+24}} = x+1 \quad /^2$

$\left(\sqrt{1+x\sqrt{x^2+24}}\right)^2 = (x+1)^2$

$\cancel{1}+x\sqrt{x^2+24} = x^2+2x+\cancel{1}$

$x\sqrt{x^2+24} = x^2+2x$

$\left(x\sqrt{x^2+24}\right)^2 = (x^2+2x)^2$

$x^2(x^2+24) = x^4+4x^3+4x^2$

$\cancel{x^4}+24x^2 = \cancel{x^4}+4x^3+4x^2$

$4x^3-20x^2=0$

$4x^2(x-5)=0$

$x=0$ OR $x=5$

check $x=0$: $\sqrt{1+0} \stackrel{?}{=} 0+1$ TRUE

check $x=5$ $\sqrt{1+5\sqrt{49}} \stackrel{?}{=} 5+1$

$\sqrt{1+5\cdot 7} = 6$ TRUE

$\therefore \boxed{x \in \{0,5\}}$

(4) $3i^3-3mi^2+(1-m)i+5 = 3i^2\cdot i-3m(-1)+(1-m)i+5$

$= -3i+3m+(1-m)i+5$

$= 3m+5+i(-3+1-m)$

$= 3m+5+i(-2-m) \in \mathbb{R}$ iff $-2-m=0$

$\boxed{m=-2}$