Math 180 Fall 2010 Name:

TEST #2 @ 180 points

Write neatly. Show all work. **Write all proofs on separate paper. Label each exercise.**

1. Prove the following property: If $f'(x)=0$ at each point x of an open interval (a,b), then $f(x)=k$ for all $x\in(a,b)$, where k is a constant.

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2. Prove the following formula: $\dfrac{d}{dx}(\cos x)=-\sin x$

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3. Find the derivative of each function and simplify as much as possible.

a) $y=\left(\dfrac{\sin t}{1+\cos t}\right)^2$

c) $f(x)=\dfrac{\sqrt{x}}{\sin\left(\sqrt{x}\right)}$

e) $r=\log_2\left(\dfrac{x^2e^2}{2\sqrt{x+1}}\right)$

b) $s=\tan^{-1}\left(x^2+1\right)\cdot\ln\left(x^2-5x+1\right)$

d) $y=q^{-2}\sin^2\left(q^3\right)$

f) $y=(x+1)^x$

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4. Find the tangent to the curve $x^2+xy-y^2=1$ at $(2,3)$.

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5. Find the critical numbers for the following function: $f(x)=\sqrt[3]{x^2-x}$

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6. Find the absolute minimum and maximum values for the following functions on the given interval:

a) $f(x)=x^2+\dfrac{2}{x},\ x\in\left[\dfrac{1}{2},4\right]$

b) $f(q)=q-2\sin q,\ q\in[0,3p]$

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7. Let $f(x)=\dfrac{1}{x}+\dfrac{1}{x^2}$. Do the following:

a) Graph the following function. Show: end behavior, behavior near vertical asymptotes (if any), intercepts, first and second derivative and their signs. Show all work and organize the information in a table, as we did in class. Label all points used.
b) What are the maximum and minimum values of the function?
c) What are the inflection points?
d) On what interval(s) is the function increasing? Decreasing?
e) On what interval(s) is the function concave up? Down?

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8) Find the following limits:

a) $\lim\limits_{x\to\infty} e^{-x}\ln x$

b) $\lim\limits_{x\to 0}\left(e^x+x\right)^{\frac{1}{x}}$

c) $\lim\limits_{q\to 0}\dfrac{\tan q-q}{q^3}$

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9. The height of an object moving vertically is given by $s=-16t^2+96t+112$, with s in feet and t in seconds. Find

a) the object's initial velocity
b) its maximum height and when it occurs
c) its velocity when $s=0$.

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10. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

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11 What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cubic centimeters?

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12. The radius r and height h of a right circular cone are related to the cone's volume V by the formula $V=\frac{1}{3}\pi r^2 h$.

a) What is dV/dt if r is constant?
b) What is dV/dt if h is constant?
c) What is dV/dt if neither r nor h is constant ?

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13. Find a linear approximation for $f(x)=(1+x)^k$ near 0, where k is a real number.

Then use the linearization to find $(1.0003)^{50}$.

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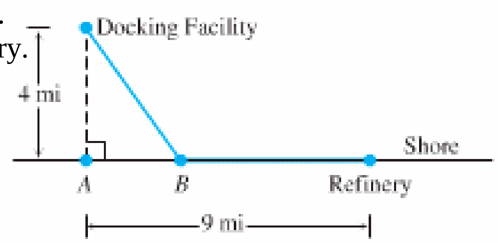
14. Supertankers off-load oil at a docking facility 4 mi offshore. The nearest refinery is 9 mi east of the shore point nearest the docking facility. A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs \$300,000 per mile if constructed underwater and \$200,000 per mile if overland. Locate point B to minimize the cost of the construction.



Extra Credit – You may choose any TWO problems.

#1@3 points Prove the following limit: $e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

#2 @ 5 points Show that $|\sin x-\cos x|\le\sqrt{2}$ for all x.

#3 @ 3 points Find a function f such that $f'(-1)=\frac{1}{2}$, $f'(0)=0$, and $f''(x)>0$ for all x, or prove that such a function cannot exist.

#4 @ 5 points For what values of c does the equation $\ln x=cx^2$ have exactly one solution?

MATH 180

TEST #2

① Given $f'(x) = 0, \forall x \in (a,b)$

Prove $f(x) = k,\ k \in \mathbb{R},\ \forall x \in (a,b)$

Proof

We need to show that

$f(x_1) = f(x_2),\ \forall x_1, x_2 \in (a,b)$

Let $x_1, x_2 \in (a,b)$; $x_1 < x_2$

Then f = differentiable on $[x_1, x_2]$ (because $f'(x)$ exists $\forall x \in [x_1, x_2]$)

f = continuous on $[x_1, x_2]$ (because any diff. fct. is cont.)

Therefore, by The Mean Value Theorem $\Rightarrow$ there is $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(x) = 0,\ \forall x \in (a,b)$ $\Rightarrow$

$f(x_2) - f(x_1) = 0 \Rightarrow f(x_1) = f(x_2)$

Therefore, if $f'(x) = 0\ \forall x$ then $f(x) = k,\ k \in \mathbb{R}$

② $\frac{d}{dx}(\cos x) = -\sin x$

Proof

Let $f(x) = \cos x$

Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h \to 0} \frac{\sin x \sin h}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\cos x) \cdot 0 - (\sin x) \cdot 1$$

$$= -\sin x$$

Therefore

$$\frac{d}{dx}(\cos x) = -\sin x$$

(3) (a) $y = \left(\frac{\sin t}{1+\cos t}\right)^2$

$y' = \frac{dy}{dt} = 2\,\frac{\sin t}{1+\cos t}\cdot\left(\frac{\sin t}{1+\cos t}\right)'$

$= \frac{2\sin t}{1+\cos t}\cdot\frac{\cos t(1+\cos t)-\sin t(-\sin t)}{(1+\cos t)^2}$

$= \frac{2\sin t}{1+\cos t}\cdot\frac{\cos t+\cos^2 t+\sin^2 t}{(1+\cos t)^2}$

$= \frac{2\sin t}{\cancel{1+\cos t}}\cdot\frac{\cancel{\cos t+1}}{(1+\cos t)^2}$

$$\boxed{y' = \frac{2\sin t}{(1+\cos t)^2}}$$

(b) $s = \tan^{-1}(x^2+1)\cdot\ln(x^2-5x+1)$

$s' = \frac{ds}{dx} = \frac{1}{1+(x^2+1)^2}(2x)\ln(x^2-5x+1) +$

$+ \tan^{-1}(x^2+1)\cdot\frac{1}{x^2-5x+1}(2x-5)$

$$\boxed{s' = \frac{2x\ln(x^2-5x+1)}{1+(x^2+1)^2} + \frac{2x-5}{x^2-5x+1}\tan^{-1}(x^2+1)}$$

(c) $f(x) = \frac{\sqrt{x}}{\sin(\sqrt{x})}$

$f'(x) = \frac{\frac{1}{2\sqrt{x}}\sin(\sqrt{x}) - \sqrt{x}\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}}{\sin^2(\sqrt{x})}$

$$\boxed{f'(x) = \frac{\sin(\sqrt{x}) - \sqrt{x}\cos(\sqrt{x})}{2\sqrt{x}\sin^2(\sqrt{x})}}$$

(d) $y = \theta^{-2}\sin^2(\theta^3)$

$y' = -2\theta^{-3}\sin^2(\theta^3) +$

$+ \theta^{-2}\,2\sin(\theta^3)\cos(\theta^3)\cdot 3\theta^2$

$y' = -2\theta^{-3}\sin^2(\theta^3) + 6\sin(\theta^3)\cos(\theta^3)$

OR

$$\boxed{y' = -2\theta^{-3}\sin^2(\theta^3) + 3\sin(2\theta^3)}$$

(e) $r = \log_2\frac{x^2e^2}{2\sqrt{x+1}}$

$r = \log_2 x^2 + \log_2 e^2 - \log_2(2\sqrt{x+1})$

$r = 2\log_2 x + 2\log_2 e - 1 - \frac{1}{2}\log_2(x+1)$

$r' = \frac{2}{x\ln 2} + 0 - 0 - \frac{1}{2(x+1)\ln 2}$

$r' = \frac{2}{x\ln 2} - \frac{1}{2(x+1)\ln 2} =$

$$\boxed{r' = \frac{3x+4}{2x(x+1)\ln 2}}$$

(f) $y = (x+1)^x$

$\ln y = \ln(x+1)^x$

$\ln y = x\ln(x+1)$

$\frac{1}{y}\cdot y' = \ln(x+1) + \frac{x}{x+1}$

$y' = y\left(\ln(x+1) + \frac{x}{x+1}\right)$

$$\boxed{y' = (x+1)^x\left(\ln(x+1) + \frac{x}{x+1}\right)}$$

(4) $x^2 + xy - y^2 = 1$

$2x + y + xy' - 2yy' = 0$

$2x + y = 2yy' - 1xy'$

$2x + y = y'(2y - x)$

$y' = \dfrac{2x+y}{2y-x}$

$m = y'\Big|_{\substack{x=2\\y=3}} = \dfrac{2(2)+3}{2(3)-2} = \dfrac{7}{4}$

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{7}{4}(x - 2)$

$\boxed{y = \frac{7}{4}x - \frac{1}{2}}$ tangent to the curve at $(2,3)$

(5) $f(x) = \sqrt[3]{x^2 - x} = (x^2 - x)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(x^2 - x)^{-\frac{2}{3}}(2x - 1)$

$f'(x) = \dfrac{2x-1}{3\sqrt[3]{(x^2-x)^2}}$

$f'(x) = 0$ iff $2x - 1 = 0$ iff $x = \frac{1}{2}$

$f'(x)$ undefined when

$x^2 - x = 0$

$x(x-1) = 0$ $\Big\langle$ $x = 0$, $x = 1$

The critical numbers are

$\boxed{x = \frac{1}{2},\ x = 0,\ x = 1}$

(6) (a) $f(x) = x^2 + \frac{2}{x}$, $[\frac{1}{2}, 4]$

1st

$f'(x) = 2x - \frac{2}{x^2} = \dfrac{2x^3 - 2}{x^2}$

$f'(x) = \dfrac{2(x^3-1)}{x^2}$

$f'(x) = 0$ iff $x = 1$

$f'(x)$ - undefined when $x = 0$

(but $0 \notin [\frac{1}{2}, 4]$

2nd

$f(\frac{1}{2}) = \frac{1}{4} + 4 = \frac{17}{4}$

$f(4) = 16 + \frac{1}{2} = \frac{33}{2}$

$f(1) = 1 + 2 = 3$

The absolute maximum is $f(4) = \frac{33}{2}$

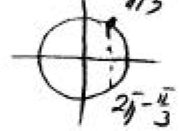
The absolute minimum is $f(1) = 3$

(6) $f(\theta) = \theta - 2\sin\theta$, $[0, 3\pi]$

1st $f'(\theta) = 1 - 2\cos\theta$

$f'(\theta)$ - defined for $\forall\theta$.

$f'(\theta) = 0$ iff $\cos\theta = \frac{1}{2}$



$\theta = \frac{\pi}{3} + 2\pi k$ or $\theta = \frac{5\pi}{3} + 2\pi k$

Therefore,

$\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$ or

$\theta = \frac{7\pi}{3}$

2nd $f(0) = 0$

$f(3\pi) = 3\pi \approx 9.4$

$f(\frac{\pi}{3}) = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - 2\frac{\sqrt{3}}{2} = \frac{\pi}{3} - \sqrt{3}$

$f(\frac{5\pi}{3}) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} - 2\left(\frac{-\sqrt{3}}{2}\right)$

$= \frac{5\pi}{3} + \sqrt{3} \approx 6.97$

$f(\frac{7\pi}{3}) = \frac{7\pi}{3} - 2\sin\frac{7\pi}{3} = \frac{7\pi}{3} - 2\frac{\sqrt{3}}{2} =$

$= \frac{7\pi}{3} - \sqrt{3} \approx 5.6$

The absolute minimum is

$f(\frac{\pi}{3}) = \frac{\pi}{3} - \sqrt{3}$

The absolute maximum is

$f(3\pi) = 3\pi$

(7) $f(x) = \frac{1}{x} + \frac{1}{x^2} = \frac{x+1}{x^2}$

x	$-\infty$	-3	-2	-1	0	∞
f'	$- - -$	$- - -$	0	$+ + +$	$\vert$	$- - - -$
f	$y=0$ H.A ↘	$-\frac{2}{9}$ ↘	$-\frac{1}{4}$ ↗	0 ↗ ∞	∞ $x=0$ V.A.	↘ $y=0$ H.A
f''	$- - -$ (concave down)	0	$+ + +$ (concave up)	$+ +$	$\vert$	$+ + +$ (concave up)

(f) • Domain $x \neq 0$

• x-int: $(-1, 0)$

• End-behavior: $\lim_{x\to\pm\infty} f(x) =$

$= \lim_{\pm\infty} \frac{x+1}{x^2} = \frac{\infty}{\infty}$ (l'Hopital)

$= \lim_{\pm\infty} \frac{1}{2x} = \frac{1}{\pm\infty} = 0 \Rightarrow y=0$ H. A.

• Behavior near $x=0$:

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x+1}{x^2} = \frac{1}{0^+} = \infty$

$\Rightarrow$ V.A. $x=0$

(f') $f'(x) = \frac{x^2 - 2x(x+1)}{x^4} = \frac{-x^2-2x}{x^4}$

$f'(x) = \frac{-x-2}{x^3}$

$f'(x) = 0$ when $x = -2$

and $f(-2) = -\frac{1}{4}$

The sign of $f'(x) = \frac{-x^2-2x}{x^4}$

is given by $y = -x^2 - 2x$ (− before −2, + between −2 and 0, − after 0)

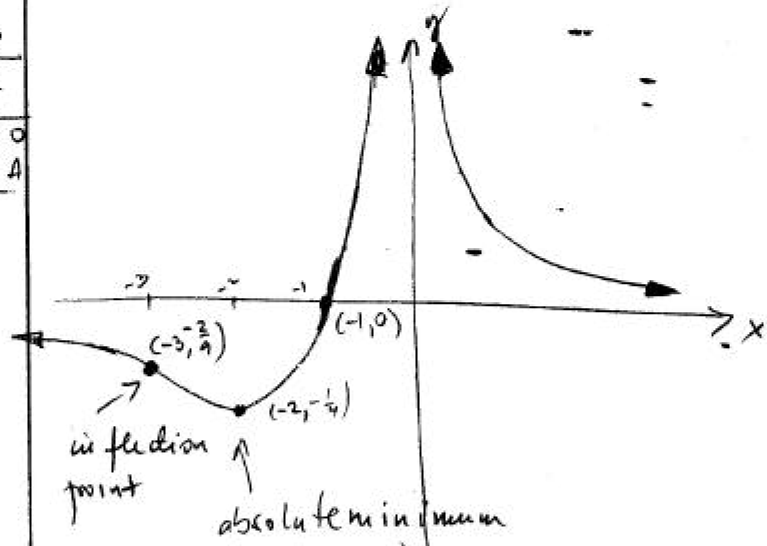
(f'') $f''(x) = \frac{-x^3 - 3x^2(-x-2)}{x^4} = \frac{2x+6}{x^4}$

$f''(x) = \frac{2(x+3)}{x^4}$

$f''(x) = 0$ when $x = -3$

The sign of f'' is given by

$y = x + 3$ (− before −3, + after −3)



b) abs. and local minimum $= f(-2) = -\frac{1}{4}$

c) inflection point $(-3, -\frac{2}{9})$

d) decreasing on $(-\infty, -2)$ and $(0, \infty)$
increasing on $[-2, 0)$

e) concave up on $(-3, 0) \cup (0, \infty)$
concave down on $(-\infty, -3)$

(8) (a) $\lim_{\infty} e^{-x}\ln x = 0\cdot\infty$

$\lim_{\infty} e^{-x}\ln x = \lim_{\infty}\frac{\ln x}{e^x} = \frac{\infty}{\infty}$

(l'Hopitol)

$= \lim_{\infty}\frac{\frac{1}{x}}{e^x} = \lim_{\infty}\frac{1}{xe^x} = \frac{1}{\infty} = 0$

so $\boxed{\lim_{\infty} e^{-x}\ln x = 0}$

(b) $\lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = 1^{\pm\infty}$

$\lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln(e^x+x)^{\frac{1}{x}}}$

$= \lim_{x\to 0} e^{\frac{1}{x}\ln(e^x+x)} = \lim_{x\to 0} e^{\frac{\ln(e^x+x)}{x}}$

$= e^{\lim_{x\to 0}\frac{\ln(e^x+x)}{x}}$

but $\lim_{x\to 0}\frac{\ln(e^x+x)}{x} = \frac{0}{0}$ (l'Hopitol)

$= \lim_{x\to 0}\frac{\frac{1}{e^x+x}(e^x+1)}{1} = \lim_{x\to 0}\frac{e^x+1}{e^x+x}$

$= \frac{2}{1} = 2$

Therefor, $\boxed{\lim_{x\to 0}(e^x+x)^{\frac{1}{x}} = e^2}$

(c) $\lim_{\theta\to 0}\frac{\tan\theta-\theta}{\theta^3} = \frac{0}{0}$

(l'Hopitol)

$= \lim_{\theta\to 0}\frac{\sec^2\theta-1}{3\theta^2} = \frac{0}{0}$

(l'Hopitol)

$= \lim_{\theta\to 0}\frac{2\sec\theta(\sec\theta\tan\theta)}{6\theta}$

$= \lim_{\theta\to 0}\frac{\sec^2\theta}{3}\cdot\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

$= \frac{1}{3}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}$

$= \frac{1}{3}\cdot 1\cdot 1 = \frac{1}{3}$

$\boxed{\lim_{\theta\to 0}\frac{\tan\theta-\theta}{\theta^3} = \frac{1}{3}}$

(9) $s = -16t^2+96t+112$:

s = height (ft)

t = time (seconds)

(a) $v(t)$ = velocity

$v(t) = \frac{ds}{dt} = -32t+96$.

$\boxed{v(0) = 96 \text{ ft/s}}$

(b) $s_{max} = ?$ $s = -16t^2+96t+112$, $t > 0$

$s' = \frac{ds}{dt} = -32t+96$

$s' = 0$ iff $t = 3$

$s'' = -32 < 0$ $\forall t > 0$.

so $s''(3) < 0$, so s is max when $t = 3$

$S_{max} = S(3) = -16(9) + 96(3) + 112$

$S_{max} = 256$ ft

when $t = 3$ seconds.

(c) $s = 0$

$-16t^2 + 96t + 112 = 0$

$t^2 - 6t - 7 = 0$

$(t-7)(t+1) = 0$

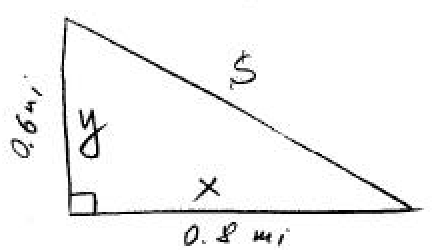
~~$t = -1$~~ or $t = 7$

$s = 0$ when $t = 7$ seconds.

$v(7) = -32(7) + 96$

$v(7) = -128$ ft/s

(10)



given: $\frac{ds}{dt} = 20$ mph

$\frac{dy}{dt} = -60$ mph

find $\frac{dx}{dt} = ?$ when $x = 0.8$ mi, $y = 0.6$ mi

Solution

$s^2 = x^2 + y^2$

$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

$x = 0.8, y = 0.6 \Rightarrow s = \sqrt{(0.8)^2 + (0.6)^2}$

$s = 1$ mi

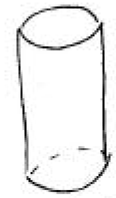
$1 \cdot (20) = 0.8\frac{dx}{dt} + 0.6(-60)$

$20 + 36 = 0.8\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{56}{0.8} = 70$ mph

The speed of the car is 70 mph

(11)



$V = 1000\ cm^3$

let r = radius of base

h = height

$V = \pi r^2 h \Rightarrow \pi r^2 h = 1000$

The can is lightest iff the total surface is min.

let S = total surface

$$\begin{cases} S = \pi r^2 + 2\pi r h \\ \pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2} \end{cases}$$

$S = \pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2}$

$S = \pi r^2 + \frac{2000}{r}, \quad r > 0$

Want to find S_{min}.

$S' = 2\pi r - \frac{2000}{r^2}$

$S' = \frac{2\pi r^3 - 2000}{r^2} = \frac{2(\pi r^3 - 1000)}{r^2}$

$S' = 0$ iff $r^3 = \frac{1000}{\pi}$ iff

$r = \frac{10}{\sqrt[3]{\pi}}$

$S'' = 2\pi + \frac{4000}{r^3} > 0 \ \forall r > 0$

so $S''\left(\frac{10}{\sqrt[3]{\pi}}\right) > 0$, therefore

S minimum when $r = \frac{10}{\sqrt[3]{\pi}}$

and $h = \frac{10}{\sqrt[3]{\pi}}$

(12) $V = \frac{1}{3}\pi r^2 h$

(a) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$

(b) $\frac{dV}{dt} = \frac{1}{3}\pi h \cdot 2r \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$

(13) $f(x) = (1+x)^k$, $k > 0$

at $x = 0$

$f'(x) = k(1+x)^{k-1}$

$f'(0) = k(1)^{k-1} = k$

$y - y_1 = m(x - x_1)$

$m = f'(0) = k$

$(x_1, y_1) = (0, 1)$

$y - 1 = k(x - 0)$

$y = 1 + kx$ the tangent to f at $(0,1)$

The linear approximation is $L(x) = 1 + kx$

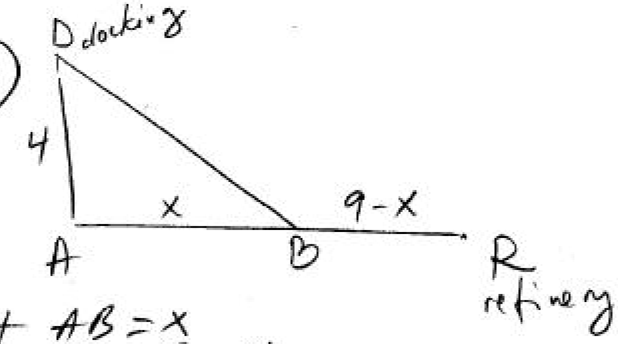
$(1+x)^k \approx 1 + kx$ near 0

$(1.0003)^{50} = (1 + 0.0003)^{50}$

$(1.0003)^{50} \approx 1 + 50(0.0003) = 1.015$

$(1.0003)^{50} \approx 1.015$

(14)



let $AB = x$

$BC = 9 - x$

Total cost $= C(x)$

$C(x) = 300{,}000\sqrt{16 + x^2} + 200{,}000(9 - x)$

where $0 \le x \le 9$

we want to find the absolute min. of $C(x)$

$C'(x) = \frac{300{,}000x}{\sqrt{16 + x^2}} - 200{,}000$

$C'(x) = 0$ iff $3x = 2\sqrt{16 + x^2}$

$9x^2 = 4x^2 + 64$

$x^2 = \frac{64}{5} \Rightarrow x = \frac{8}{\sqrt{5}} \approx 3.58$ mi

$C(0) = 3{,}000{,}000$ \$

$C(9) = 2{,}954{,}660$ \$

$C(3.58) = 2{,}694{,}430$ \$ minimum cost when $AB = 3.58$ miles

EXTRA CREDIT

(1) see note a textbook

(2) $|\sin x - \cos x| \le \sqrt{2}$

Proof

let $f(x) = \sin x - \cos x$

let $x \in [0, 2\pi]$ - one period of $f(x)$

x	0		$3\pi/4$		$7\pi/4$		2π
f'	+ + +		0	- - -	0	+ + +	
f	-1	$\nearrow$	$\sqrt{2}$	$\searrow$	$-\sqrt{2}$	$\nearrow$	-1

$f(0) = -1, \quad f(2\pi) = -1$

$f'(x) = \cos x + \sin x$

$f'(x) = 0$ iff $\sin x = -\cos x$ $\quad \Big/ \div \cos x \ne 0$

note that $\cos x \ne 0$

iff $\tan x = -1$

iff $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

Sign of f': Test point $x = 0$, $f'(0) > 0$

$x = \frac{3\pi}{2}$, $f'(\frac{3\pi}{2}) < 0$

$x = 2\pi$, $f'(\frac{2\pi}{2}) > 0$

$f(\frac{3\pi}{4}) = \sqrt{2}, \quad f(\frac{7\pi}{4}) = -\sqrt{2}$

Note that abs. max is $\sqrt{2}$
abs. min is $-\sqrt{2}$

Therefore, $-\sqrt{2} \le f(x) \le \sqrt{2}$

$|\sin x - \cos x| \le \sqrt{2}$

(3) $f'(-1) = \frac{1}{2}, \quad f'(0) = 0$

$f''(x) > 0 \Rightarrow f'(x)$ is increasing

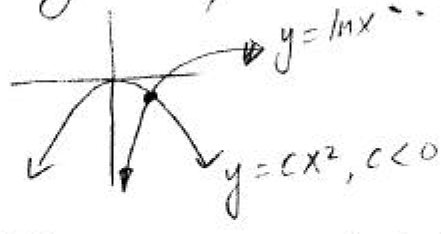
$\Rightarrow f'(-1) < f'(0)$

but $f'(-1) = \frac{1}{2}$ and $f'(0) = 0$

Contradiction!

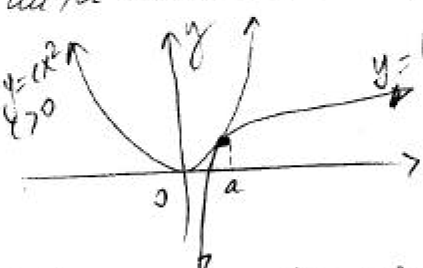
No such function exists.

(4) Note that if $c < 0$, the curve $y = cx^2$ (parabola with vertex (0,0) opening down) and the curve $y = \ln x$ have exactly one point in common



If $c = 0$, then $y = \ln x$ and $y = 0$ (x-axis) have also one point in common.

If $c > 0$, $y = cx^2$ is a parabola with vertex (0,0), opening up



let's assume the common point has $x = a \Rightarrow \ln a = ca^2$

The curves also have a common tangent at $x = a$ (same slope) $\Rightarrow \frac{1}{a} = 2ca$

$$\begin{cases} \ln a = ca^2 \\ \frac{1}{a} = 2ca \Rightarrow a^2 = \frac{1}{2c} \end{cases}$$

$\Rightarrow \ln a = \frac{1}{2} \Rightarrow a = e^{\frac{1}{2}}$

Then $c = \frac{\ln a}{a^2} = \frac{\ln e^{\frac{1}{2}}}{e} = \frac{1}{2e}$

Therefore, $\ln x = cx^2$ has exactly one sol. when $c \le 0$ or $c = \frac{1}{2e}$