Math 180 Fall 2010 Name:

QUIZ #3 @ 80 points

Write neatly. Show all work. **Write all responses on separate paper.** **Clearly label the exercises.**

1. Graph the given function $f(x)=3x^2$ between $x=0$ and $x=1$. Then:
   a) Using four rectangles whose height is given by the value of the function at the right-end of each subinterval, estimate the area under the graph.
   b) Find the exact area between the graph and the x-axis.

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2. Graph the functions $y=2\sin x$ and $y=\sin 2x$ when $0\le x\le \pi$, then find the area of the region enclosed by the curves .

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3. Find the total area between the region and the x-axis: $y=x^3-3x^2+2x,\quad 0\le x\le 2$

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4. If $\int_1^x f(t)\,dt = x^2-3x+5$, find $f(x)$.

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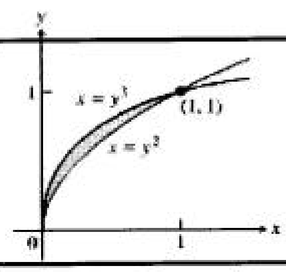
5. Find the total area of the shaded region between $x=y^3$ and $x=y^2$.



Find the following:

6) $\int\left(e^{2x}+x^5-4^x+\dfrac{1}{x}\right)dx$

7) $\int\left(2^{t+1}-t^{\sqrt{2}}\right)dt$

8) $\int\cos\theta\left(\tan\theta+\sec\theta\right)d\theta$

9) $\int_0^{\pi}5\left(5-4\cos t\right)^{1/4}\sin t\,dt$

10) $\int\dfrac{\sin 2t}{1+\sin^2 t}dt$

11) $\int e^x\sin x\,dx$

12) $\int_1^2\ln x\,dx$

13) $\int\tan\dfrac{x}{2}\,dx$

14) $\int\cos^2 x\,dx$

15) $\int\dfrac{x}{\sqrt{1-4x^2}}dx$

16) $\int_1^4\dfrac{dx}{2\sqrt{x}\left(1+\sqrt{x}\right)^2}$

17) $\int_0^{\pi/3}\theta^2\sin\theta\,d\theta$

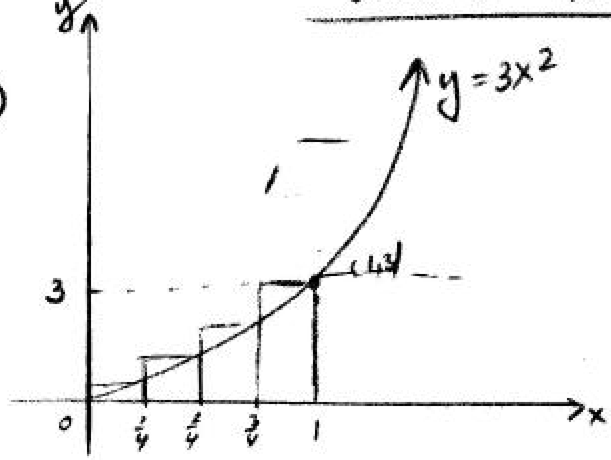
18) $\int_0^1\dfrac{8r}{4r^2-5}dr$

19) $\int_0^{2\pi}\dfrac{\cos x}{\sqrt{4+3\sin x}}dx$

20) $\int\dfrac{5}{9+4r^2}dr$

Quiz 3 - Solutions

(1)



(a) let A = area under $y = 3x^2$ and x-axis

$$A \approx \sum_{k=1}^{4} f(c_k)\,\Delta x \text{, where}$$

$$\Delta x = \frac{1}{4}$$

$$c_k = \frac{k}{4}, \quad k = \overline{1,4}$$

$$\sum_{k=1}^{4} f(c_k)\,\Delta x = \sum_{k=1}^{4} 3c_k^2 \cdot \frac{1}{4} = \frac{3}{4}\sum_{k=1}^{4}\left(\frac{k}{4}\right)^2$$

$$= \frac{3}{4}\cdot\frac{1}{16}\sum_{k=1}^{4} k^2 = \frac{3}{4\cdot 16}\cdot\frac{4(4+1)(2\cdot 4+1)}{6}$$

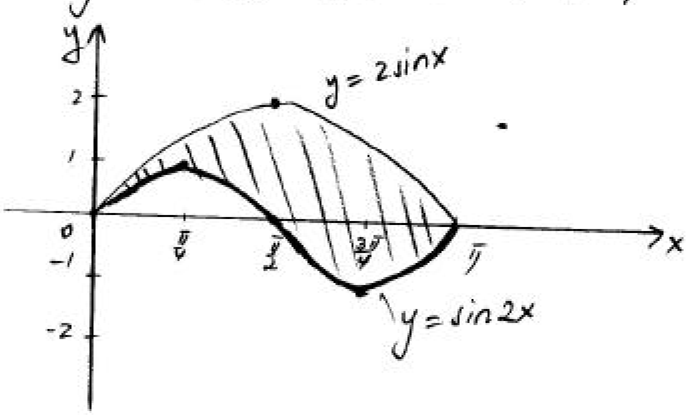
$$= \frac{5\cdot 9}{16\cdot 2} = \frac{45}{32} \Rightarrow \boxed{A \approx \frac{45}{32}}$$

(b) $A = \int_0^1 f(x)\,dx = \int_0^1 3x^2\,dx$

$$= x^3\Big]_0^1 = 1^3 - 0 = 1$$

$$\boxed{A = 1}$$

(2) $y = 2\sin x$ has $A = 2,\ T = 2\pi$
$y = \sin 2x$ has $A = 1,\ T = \pi$



let A = area between the curves

$$A = \int_0^{\pi}(2\sin x - \sin 2x)\,dx$$

$$A = -2\cos x\Big]_0^{\pi} - \left(-\frac{1}{2}\cos 2x\Big]_0^{\pi}\right)$$

$$A = -2(\cos\pi - \cos 0) + \frac{1}{2}(\cos 2\pi - \cos 0)$$

$$A = -2(-2) + \frac{1}{2}(0) \Rightarrow \boxed{A = 4}$$

(3) $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

Find X-I:

$$x^3 - 3x^2 + 2x = 0$$

$$x(x^2 - 3x + 2) = 0$$

$$x(x-1)(x-2) = 0$$

$$x = 0, \quad x = 1, \quad x = 2$$

0 1 2

let A = area between the curve and x-axis

$$A = \left|\int_0^1 y\,dx\right| + \left|\int_1^2 y\,dx\right|$$

$$A = \left|\int_0^1 (x^3 - 3x^2 + 2x)\,dx\right| + \left|\int_1^2 (x^3 - 3x^2 + 2x)\,dx\right|$$

$$A = \left|\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1\right| + \left|\left[\frac{x^4}{4} - x^3 + x^2\right]_1^2\right|$$

$$A = \left|\frac{1}{4} - 1 + 1\right| + \left|\frac{2^4}{4} - 2^3 + 2^2 - \left(\frac{1}{4} - 1 + 1\right)\right|$$

$$A = \left|\frac{1}{4}\right| + \left|4 - 8 + 4 - \frac{1}{4}\right|$$

$$A = \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\boxed{A = \frac{1}{2}}$$

(4) $\int_1^x f(t)\,dt = x^2-3x+5$

Then $\frac{d}{dx}\int_1^x f(t)dt = \frac{d}{dx}(x^2-3x+5)$

$\boxed{f(x) = 2x-3}$

(5) Let A = shaded area
we'll find A by integrating with respect to y (using horizontal approximating □'s)

$A = \int_0^1 (y^2-y^3)\,dy = \left[\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1$

$A = \frac{1}{3}-\frac{1}{4} = \frac{1}{12} \Rightarrow \boxed{A = \frac{1}{12}}$

(6) $\int\left(e^{2x}+x^5-4^x+\frac{1}{x}\right)dx =$

$= \frac{e^{2x}}{2}+\frac{x^6}{6}-\frac{4^x}{\ln 4}+\ln|x|+C,$
$C\in\mathbb{R}$

(7) $\int\left(2^{t+1}-t^{\sqrt{2}}\right)dt =$

$= \frac{2^{t+1}}{\ln 2}-\frac{t^{\sqrt{2}+1}}{\sqrt{2}+1}+C,\ C\in\mathbb{R}$

(8) $\int \cos\theta(\tan\theta+\sec\theta)\,d\theta =$

$\int(\sin\theta+1)\,d\theta = -\cos\theta+\theta+C,$
$C\in\mathbb{R}$

(9) $\int_0^{\pi} 5(5-4\cos t)^{\frac{1}{4}}\sin t\,dt$

let $5-4\cos t = u$
then $4\sin t\,dt = du$
$\sin t\,dt = \frac{1}{4}du$

when $t=0,\ u=1$
$t=\pi,\ u=9$

Then, $\int_0^{\pi} 5(5-4\cos t)^{\frac{1}{4}}\sin t\,dt =$

$= \int_1^9 5\,u^{\frac{1}{4}}\cdot\frac{1}{4}\,du = \frac{5}{4}\int_1^9 u^{\frac{1}{4}}\,du$

$= \frac{5}{4}\left.\frac{u^{\frac{1}{4}+1}}{\frac{5}{4}}\right]_1^9 = \left.u^{\frac{5}{4}}\right]_1^9 =$

$= 9^{\frac{5}{4}}-1 = (3^2)^{\frac{5}{4}}-1 = 3^{\frac{5}{2}}-1$

(10) $\int\frac{\sin 2t}{1+\sin^2 t}\,dt$

let $1+\sin^2 t = u > 0$
then $2\sin t\cos t\,dt = du$
$\sin 2t\,dt = du$

Then, $\int\frac{\sin 2t}{1+\sin^2 t}\,dt =$

$= \int\frac{du}{u} = \ln|u|+C$

$= \ln|1+\sin^2 t|+C$

$= \ln(1+\sin^2 t)+C,\ C\in\mathbb{R}$

(11) let $I = \int e^x\sin x\,dx$
we'll use integration by Parts:

let $f = e^x$ $g' = \sin x$ (or viceversa)
then $f' = e^x$ $g = -\cos x$

Then, $I = -e^x\cos x - \int -e^x\cos x\,dx$

$I = -e^x\cos x + \int e^x\cos x\,dx$

we'll use integration by Parts.

let $f = e^x$ $g' = \cos x$
then $f' = e^x$ $g = \sin x$

Then, $i = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\, dx\right)$

$i = -e^x \cos x + e^x \sin x - i$

$2i = -e^x \cos x + e^x \sin x$

$i = \frac{e^x}{2}(-\cos x + \sin x) + C$, $C \in \mathbb{R}$

(12) $\int^{2} \ln x\, dx$

We'll use integration by Parts:

$\left[\begin{array}{l} \text{let } f = \ln x \searrow g' = 1 \\ \text{then } f' = \frac{1}{x} \leftarrow g = x \end{array}\right.$

Then, $\int_1^2 \ln x\, dx = x \ln x \Big]_1^2 - \int_1^2 x \cdot \frac{1}{x}\, dx$

$= 2\ln 2 - 1 \ln 1 - \int_1^2 dx$

$= 2\ln 2 - x\Big]_1^2 = \ln 4 - (2-1)$

$= \ln 4 - 1$

(13) $\int \tan \frac{x}{2}\, dx = \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}\, dx$

let $\cos \frac{x}{2} = u$

then $-\frac{1}{2} \sin \frac{x}{2}\, dx = du$

$\sin \frac{x}{2}\, dx = -2\, du$

Then, $\int \tan \frac{x}{2}\, dx =$

$= \int \frac{1}{u}(-2)\, du = -2 \int \frac{1}{u}\, du$

$= -2 \ln|u| + C = -2 \ln|\cos \frac{x}{2}| + C$, $C \in \mathbb{R}$

(14) Let $i = \int \cos^2 x\, dx$

Method I - integration by Parts

$\left[\begin{array}{l} \text{let } f = \cos x \searrow g' = \cos x \\ \text{then } f' = -\sin x \leftarrow g = \sin x \end{array}\right.$

Then, $i = \sin x \cos x - \int -\sin^2 x\, dx$

$i = \sin x \cos x + \int (1 - \cos^2 x)\, dx$

$i = \frac{\sin 2x}{2} + x - i$

$2i = \frac{\sin 2x}{2} + x$

$i = \frac{\sin 2x}{4} + \frac{x}{2} + C, \quad C \in \mathbb{R}$

Method II $\cos 2x = 2\cos^2 x - 1$

so $\cos^2 x = \frac{1 + \cos 2x}{2}$

Then $i = \int \frac{1 + \cos 2x}{2}\, dx$

$i = \int \frac{1}{2}\, dx + \int \frac{\cos 2x}{2}\, dx$

$i = \frac{1}{2} x + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C$, $C \in \mathbb{R}$

(15) $\int \frac{x}{\sqrt{1 - 4x^2}}\, dx$

$\left[\begin{array}{l} \text{Let } \sqrt{1-4x^2} = u \\ \text{then } \frac{1}{2\sqrt{1-4x^2}}(-8x)\, dx = du \\ \qquad \frac{x}{\sqrt{1-4x^2}}\, dx = -\frac{1}{4}\, du \end{array}\right.$

Then, $\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int -\frac{1}{4}\, du$

$= -\frac{1}{4} u + C = -\frac{1}{4}\sqrt{1-4x^2} + C$, $C \in \mathbb{R}$

(OR use $1 - 4x^2 = u$)

(16) $\displaystyle\int_1^4 \frac{dx}{2\sqrt{x}\,(1+\sqrt{x})^2} = I$

$\left[\begin{array}{l} \text{let } 1+\sqrt{x} = u \\ \text{then } \frac{1}{2\sqrt{x}}\,dx = du \\ \text{when } x=1,\ u=2 \\ \qquad\ \ x=4,\ u=3 \end{array}\right.$

Then, $\displaystyle I = \int_2^3 \frac{du}{u^2} = -u^{-1}\Big]_2^3$

$I = -(3^{-1}-2^{-1}) = -\left(\frac{1}{3}-\frac{1}{2}\right) = \frac{1}{6}$

(17) $\displaystyle\int_0^{\pi/3} \theta^2 \sin\theta\, d\theta = I$

We'll use integration by parts:

$\left[\begin{array}{l} \text{let } f=\theta^2 \searrow g' = \sin\theta \\ \text{then } f' = 2\theta \leftarrow g = -\cos\theta \end{array}\right.$

Then, $\displaystyle I = -\theta^2\cos\theta\Big]_0^{\pi/3} - \int_0^{\pi/3} -2\theta\cos\theta\, d\theta$

$\displaystyle I = -\left(\frac{\pi^2}{9}\cos\frac{\pi}{3} - 0\right) + 2\int_0^{\pi/3} \theta\cos\theta\, d\theta$

We'll use Parts again

$\left[\begin{array}{l} \text{let } f=\theta \searrow g' = \cos\theta \\ \text{then } f' = 1 \leftarrow g = \sin\theta \end{array}\right.$

$\displaystyle I = \frac{-\pi^2}{9}\cdot\frac{1}{2} + 2\left(\theta\sin\theta\Big]_0^{\pi/3} - \int_0^{\pi/3}\sin\theta\, d\theta\right)$

$\displaystyle = \frac{-\pi^2}{18} + 2\left(\frac{\pi}{3}\sin\frac{\pi}{3} - 0 - \left(-\cos\theta\Big]_0^{\pi/3}\right)\right)$

$\displaystyle = \frac{-\pi^2}{18} + \frac{2\pi}{3}\frac{\sqrt{3}}{2} + \left(\cos\frac{\pi}{3} - \cos 0\right)$

$\displaystyle = \frac{-\pi^2}{18} + \frac{\pi\sqrt{3}}{3} + \frac{1}{2} - 1 = \frac{-\pi^2}{18} + \frac{\pi\sqrt{3}}{3} - \frac{1}{2}.$

(18) $\displaystyle\int_0^1 \frac{8r}{4r^2-5}\, dr$

$\left[\begin{array}{l} \text{let } 4r^2-5 = u \\ \text{then } 8r\,dr = du \\ \text{when } r=0,\ u=-5 \\ \qquad\ \ r=1,\ u=-1 \end{array}\right.$

Then,

$\displaystyle\int_0^1 \frac{8r}{4r^2-5}\, dr = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$

$= \ln 1 - \ln 5 = -\ln 5$

(19) $\displaystyle\int_0^{2\pi} \frac{\cos x}{\sqrt{4+3\sin x}}\, dx = I$

let $\sqrt{4+3\sin x} = u$

then $\displaystyle\frac{3\cos x\, dx}{2\sqrt{4+3\sin x}} = du$

$\displaystyle\frac{\cos x\, dx}{\sqrt{4+3\sin x}} = \frac{2}{3}\,du$

when $x=0,\ u=2$

$x=2\pi,\ u=2$

Then, $\displaystyle I = \int_2^2 \frac{2}{3}\, du = 0$

(or use $4+3\sin x = u$)

(20) $\displaystyle\int \frac{5}{9+4r^2}\, dr =$

$\displaystyle = \int \frac{5}{9(1+\frac{4}{9}r^2)}\, dr = \int \frac{5}{9(1+(\frac{2r}{3})^2)}\, dr$

$\displaystyle I = \frac{5}{9}\int \frac{1}{1+(\frac{2r}{3})^2}\, dr$

$$\left[\begin{array}{l} \text{let } \frac{2r}{3} = u \\ \text{then } \frac{2}{3}\,dr = du \Rightarrow dr = \frac{3}{2}\,du \end{array}\right.$$

Then, $I = \frac{5}{9}\int \frac{1}{1+u^2}\cdot\frac{3}{2}\,du$

$I = \frac{5}{9}\cdot\frac{3}{2}\int\frac{1}{1+u^2}\,du$

$I = \frac{5}{6}\tan^{-1}u + C$

$I = \frac{5}{6}\tan^{-1}\left(\frac{2r}{3}\right) + C,\ C \in \mathbb{R}$

EXTRA CREDIT

(1) $\displaystyle\int_{-1}^{-\frac{\sqrt{2}}{2}} \frac{dy}{y\sqrt{4y^2-1}} = I$

$$\left[\begin{array}{l} \text{let } 2y = u \Rightarrow y = \frac{u}{2} \\ \text{then } 2\,dy = du,\ dy = \frac{1}{2}\,du \\ \text{when } y=-1,\ u=-2 \\ \qquad\quad y = -\frac{\sqrt{2}}{2},\ u = -\sqrt{2} \end{array}\right.$$

Then, $\displaystyle I = \int_{-2}^{-\sqrt{2}} \frac{du}{\frac{u}{2}\sqrt{u^2-1}}\cdot\frac{1}{2}$

$\displaystyle I = \int_{-2}^{-\sqrt{2}} \frac{du}{u\sqrt{u^2-1}}$

note that $u<0$, so $|u| = -u$

$\displaystyle I = \int_{-2}^{-\sqrt{2}} \frac{-du}{|u|\sqrt{u^2-1}} = -\sec^{-1}u\Big]_{-2}^{-\sqrt{2}}$

$= -\left(\sec^{-1}(-\sqrt{2}) - \sec^{-1}(-2)\right)$

$= -\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right) = \frac{-\pi}{12}$

(2) $I = \int x\sqrt{1-x}\,dx$

$$\left[\begin{array}{l} \text{let } \sqrt{1-x} = u \\ \text{then } \frac{-1}{2\sqrt{1-x}}\,dx = du \\ \qquad dx = -2u\,du \\ \text{also, } 1-x = u^2 \Rightarrow x = 1-u^2 \end{array}\right.$$

Then, $I = \int (1-u^2)\cdot u\cdot(-2u)\,du$

$I = -2\int(1-u^2)u^2\,du$

$I = -2\int(u^2-u^4)\,du$

$I = 2\int(u^4-u^2)\,du = \frac{2u^5}{5} - \frac{2u^3}{3} + C$

$I = \frac{2}{5}\sqrt{(1-x)^5} - \frac{2}{3}\sqrt{(1-x)^3} + C$

$C \in \mathbb{R}$

(OR use $1-x=u$
OR use integration by parts
with $f = x$, $g' = \sqrt{1-x}$)