TEST #1 @ 145 points

Write neatly. Show all work. **Write all proofs on separate green paper. Label each exercise.**

PART I @ 10 points – open notebook

1. Prove the following:

If a function f is differentiable at a point a, then it is continuous at a.

2. Prove the following:

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

3. Prove the following:

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

4. Complete the following formulas. Do not prove.

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Write neatly. Show all work. **Write all proofs on separate white paper. Label each exercise.**

PART II @ 135 points

1. a) Write the definition of

$$\lim_{x\to a} g(x) = L.$$

b) Write the mathematical definition of:

The function $y = f(x)$ *is continuous at* $x = a$.

c) When is a function $y = f(x)$ discontinuous at $x = a$?

---

2. Prove that $\lim_{x\to 0} x^4 \sin\dfrac{3}{x} = 0$.

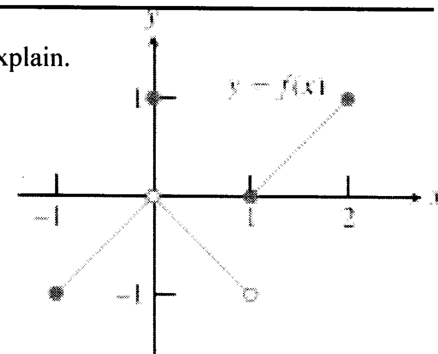
---

3. Prove the following formula:

$$\frac{d}{dx}(\sin x) = \cos x$$

---

4. Use the following graph to answer the questions. If a limit does not exist, explain.



a) $\lim_{x\to 0} f(x)$

b) $\lim_{x\to 1} f(x)$

c) Is the function f continuous at $x = 0$? Explain using the definition.

d) At what points in the interval $(-1, 2)$ is the function discontinuous? Explain.

---

5. Let $f(x) = \begin{cases} \dfrac{x+1}{x+2} & \text{if } x \neq 3 \\ \text{k} & \text{if } x = 3 \end{cases}$. Answer the following:

a) Find k such that f is continuous at $x = 3$. Do not just write an answer, show a correct mathematical proof.

b) Give a value for k for which the function is not differentiable. Explain how you know that value will make f not differentiable at $x = 3$.

---

6. Find an equation for the tangent line to the curve $y = \ln(3x)$ at $x = \dfrac{e}{3}$.

---

7. Use the definition of derivative to find the following:

a) $f'(0)$ if $f(x) = x^2 + 3x - 1$.

b) $\dfrac{dg}{dx}$ if $g(x) = \sqrt{2x+1}$.

8. At what points does the graph of $f(x)=x^4+\frac{11}{3}x^3-\frac{3}{2}x^2+5$ have a horizontal tangent? Do not just write an answer, prove and explain.

---

9. Let $f(x)=\begin{cases} x^3\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$.

Prove whether the function is differentiable at $x=0$ or not.

---

10. Find the following limits. Do not just write an answer, show proof.

a) $\lim\limits_{x\to 5^+}\frac{(x+1)(5-x)}{|x-5|}$

d) $\lim\limits_{x\to 1}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

b) $\lim\limits_{\theta\to 0}\frac{2\theta+4\theta\cos\theta}{\sin\theta\cos\theta}$

e) $\lim\limits_{x\to \infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

---

11. a) Use implicit differentiation to find $\frac{dy}{dx}$ if $x^3+y^3=5xy$.

b) At what point(s) is the tangent to the curve horizontal ? Simplify as much as possible and give the coordinates of the point(s).

---

12. Calculate the following and simplify as much as possible. You may use logarithmic differentiation if you find it necessary.

a) $g'(x)$ if $g(x)=2\left(\frac{2}{\sqrt{x}}+1\right)^{-\frac{1}{3}}$

b) $\frac{dy}{dt}$ if $y=\ln\frac{1}{t\sqrt{t+1}}$

c) $h''(x)$ - the second derivative if $h(x)=\cot\left(x^2\right)$

d) $f'(x)$ if $f(x)=\sin^{-1}\left(\cos(2x+5)\right)$

e) $y'(x)$ if $y=x^{x+1}$

---

13. Water is flowing at a constant rate into a cylindrical tank. Let $V(t)$ be the volume of water in the tank and $h(t)$ be the height of water in the tank at time t.

a) What are the meanings of $V'(t)$ and $h'(t)$? Are these derivatives positive, negative, or zero? Explain.

b) Is $V''(t)$ positive, negative, or zero? Explain.

c) Assume the radius of the base is 10 m and the water is flowing into the tank at a rate 250 L/min. How rapidly will the water level inside the tank rise? (Note: $1m^3 = 1000L$ water.)

---

14. The position of a particle is given by the equation

$$s(t) = 2t^3 - 5t^2 + t$$

where t is measured in seconds and s in meters

a) Find the velocity at time $t = 2$ seconds.

b) When is the particle at rest?

c) Find the acceleration at time $t = 2$ seconds.

EXTRA CREDIT – You may choose only TWO exercises for extra credit.

#1 @ 2 points

When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius *R* and length *l*. Because of friction at the walls of the tube, the velocity *v* of the blood is greatest along the central axis of the tube and decreases as the distance *r* from the axis increases until *v* becomes 0 at the wall. The relationship between *v* and *r* is given by

$v = \dfrac{P}{4nl}\left(R^2 - r^2\right)$, where *n* is the viscosity of the blood and *P* is the pressure difference between the ends of the tube. If *P* and *l* are constant, then *v* is a function of *r*. Find $\dfrac{dv}{dr}$.

#2 @ 3 points

For what positive values of *c* do the graphs of the two given functions have exactly one point in common?

$$f(x) = \ln x,\; g(x) = cx^2.$$

#3 @ 4 points

Find numbers *a* and *b* such that $\lim\limits_{x\to 0} \dfrac{\sqrt{ax+b} - 2}{x} = 1$.

180 Fall 08

TEST 1 - SOLUTIONS

PART I

(1) Given: f - diff. at $x=c$

Prove: f - cont. at $x=c$

(will prove that $\lim_{x\to c} f(x) = f(c)$)

Proof

f diff. at $x=c$ =>

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \in \mathbb{R}$$

let $f(x)-f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c)$

true $\forall x \neq c$

$$\lim_{x\to c} (f(x)-f(c)) = \lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right)$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x-c)$$

$$= f'(c) \cdot 0 = 0$$

so, $\lim_{x\to c}(f(x)-f(c)) = 0$

$\lim_{x\to c} f(x) - f(c) = 0$

$\lim_{x\to c} f(x) = f(c)$

Therefore, f is continuous at $x=c$.

(2) Prove $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Proof

Know $\cos 2a = 1 - 2\sin^2 a$

so $\cos x = 1 - 2\sin^2 \frac{x}{2}$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{x}$$

$$= -2 \lim_{x\to 0} \frac{\sin^2 \frac{x}{2}}{x}$$

(let $\frac{x}{2} = \theta$; when $x \to 0$, $\theta \to 0$)

$$= -2 \lim_{\theta\to 0}\left(\frac{\sin\theta}{2\theta} \cdot \sin\theta\right)$$

$$= \frac{-2}{2} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta\to 0} \sin\theta$$

$$= -1 \cdot 1 \cdot 0 = 0$$

(3) $\frac{d}{dx}(a^x) = a^x \ln a$

Proof

let $a = e^{\ln a}$

then $a^x = (e^{\ln a})^x$

$a^x = e^{x \ln a}$ $\Big/ \frac{d}{dx}$

$\frac{d}{dx} a^x = (a^x)' = (e^{x\ln a})'$

(Note: $(e^u)' = e^u \cdot u'$ with $u = x\ln a$)

$$= e^{x\ln a}(x\ln a)'$$

$$= e^{x\ln a} \cdot \ln a$$

$$= e^{\ln a^x} \cdot \ln a$$

$$= a^x \cdot \ln a$$

so, $\frac{d}{dx}(a^x) = a^x \ln a$

(4) $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=a}}$$, where $f(a)=b$

PART II

(1) (a) $\lim_{x\to a} g(x) = L$ iff

the values of $g(x)$ get arbitrarily close to L by taking x to be sufficiently close to a (on either side of a), but not equal to a.

(b) f continuous at $x=a$ iff

$\lim_{x\to a} f(x) = f(a)$

(c) f discontinuous at $x=a$ if

1° - $f(a)$ is not defined

OR

2° - $\lim_{x\to a} f(x)$ doesn't exist

or

3° - $\lim_{x\to a} f(x) \neq f(a)$

---

(2) Prove $\lim_{x\to 0} x^4 \sin\frac{3}{x} = 0$

Proof

$-1 \le \sin\frac{3}{x} \le 1$, any $x \neq 0$

Multiply each side by $x^4 > 0$

$-x^4 \le x^4 \sin\frac{3}{x} \le x^4$

$\lim_{x\to 0} x^4 = \lim_{x\to 0}(-x^4) = 0$

By Squeeze (Sandwich) Theorem

$\Rightarrow \lim_{x\to 0} x^4 \sin\frac{3}{x} = 0$

(3) Prove

$$\frac{d}{dx}(\sin x) = \cos x$$

Proof

let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sin x(\cos h - 1)}{h} + \frac{\sin h\cos x}{h}\right)$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\sin h\cos x}{h}$$

$$= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}$$

$$= (\sin x)\cdot 0 + \cos x\cdot(1) = \cos x$$

Therefore, $f'(x) = \frac{d}{dx}(\sin x) = \cos x$

---

(4) (a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to 1} f(x)$ doesn't exist

because $\lim_{x\to 1^-} f(x) = -1$ and

$\lim_{x\to 1^+} f(x) = 0$

(c) f is not continuous at 0

because $\lim_{x\to 0} f(x) \neq f(0)$

$\lim_{x\to 0} f(x) = 0$, while $f(0) = 1$

(d) f is discontinuous at $x=0$ and $x=1$

At $x=0$ because
$$\lim_{x\to 0} f(x) \neq f(0)$$

At $x=1$ because
$$\lim_{x\to 1} f(x) \text{ doesn't exist}$$

---

(5) (a) f continuous at $x=3$ iff
$$\lim_{x\to 3} f(x) = f(3)$$

$$\left.\begin{array}{l} f(3)=k \\ \lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x+1}{x+2} = \frac{4}{5} \end{array}\right\} \Rightarrow$$

$\Rightarrow$ in order for f to be continuous we need
$$k = \frac{4}{5}$$

(b) Any $k \neq \frac{4}{5}$

If $k \neq \frac{4}{5}$, f will not be continuous at $x=3$, therefore f will not be differentiable at $x=3$

(6) $y = \ln(3x)$, $x = \frac{e}{3}$

Solution

For the equation of the tangent line we need the point of tangency and the slope.

point: when $x = \frac{e}{3}$,
$$y = \ln\left(3 \cdot \frac{e}{3}\right) = \ln e = 1$$
$$\left(\frac{e}{3}, 1\right)$$

slope: $\frac{dy}{dx} = \frac{1}{3x}(3x)' = \frac{1}{x}$

so $m = \left.\frac{dy}{dx}\right|_{x=\frac{e}{3}} = \frac{1}{\frac{e}{3}} = \frac{3}{e}$

$$m = \frac{3}{e}$$

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \frac{3}{e}\left(x - \frac{e}{3}\right)$$
$$y - 1 = \frac{3}{e}x - 1$$
$$y = \frac{3}{e}x$$ the tangent line to the graph of $y = \ln(3x)$ at $\left(\frac{e}{3}, 1\right)$

(7)

(a) $f(x) = x^2 + 3x - 1$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{(x^2 + 3x - 1) - (-1)}{x}$$

$$= \lim_{x \to 0} \frac{x^2 + 3x}{x} = \lim_{x \to 0} \frac{\cancel{x}(x+3)}{\cancel{x}}$$

$$= \lim_{x \to 0} (x+3) = 3$$

so $f'(0) = 3$

(b) $g(x) = \sqrt{2x+1}$

$$\frac{dg}{dx} = g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{2(x+h)+1} - \sqrt{2x+1}\right)\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{(2(x+h)+1) - (2x+1)}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h + \cancel{1} - \cancel{2x} - \cancel{1}}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \frac{2}{\sqrt{2x+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}} = g'(x)$$

(8) $f(x) = x^4 + \frac{11}{3}x^3 - \frac{3}{2}x^2 + 5$

has a horizontal tangent iff the slope of the tangent is zero

$$f'(x) = 4x^3 + \frac{11}{3} \cdot 3x^2 - \frac{3}{2} \cdot 2x$$

$$f'(x) = 4x^3 + 11x^2 - 3x$$

$$f'(x) = 0 \Rightarrow x(4x^2 + 11x - 3) = 0$$

$x = 0$ OR

$4x^2 + 11x - 3 = 0$

$$x = \frac{-11 \pm \sqrt{169}}{8} = \frac{-11 \pm 13}{8} \begin{cases} \frac{1}{4} \\ -3 \end{cases}$$

Therefore, the graph will have a horizontal tangent when $x = 0$, $x = \frac{1}{4}$, $x = -3$.

(9) f is differentiable at $x = 0$ iff $f'(0) \in \mathbb{R}$ ($f'(0)$ exists)

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^3 \sin\frac{1}{x} - 0}{x}$$

$$= \lim_{x \to 0} x^2 \sin\frac{1}{x}$$

$$= 0 \in \mathbb{R}$$

so f is differentiable at $x = 0$

(10)

(a) $\displaystyle\lim_{x\to 5^+} \frac{(x+1)(5-x)}{|x-5|} =$

Note. $|x-5| = \begin{cases} x-5 & \text{if } x \geq 5 \\ 5-x & \text{if } x<5 \end{cases}$

when $x \to 5^+$, $x > 5$, so $|x-5| = x-5$

$\displaystyle = \lim_{x\to 5^+} \frac{(x+1)(5-x)}{x-5}$

$\displaystyle = \lim_{x\to 5^+} -(x+1) = -(5+1) = -6$

so $\displaystyle\lim_{x\to 5^+} \frac{(x+1)(5-x)}{|x-5|} = -6$

(b) $\displaystyle\lim_{\theta\to 0} \frac{2\theta + 4\theta\cos\theta}{\sin\theta\cos\theta} =$

$\displaystyle = \lim_{\theta\to 0} \frac{2\theta(1+2\cos\theta)}{\sin\theta\cos\theta}$

$\displaystyle = \lim_{\theta\to 0} \frac{2\theta}{\sin\theta} \cdot \lim_{\theta\to 0} \frac{1+2\cos\theta}{\cos\theta}$

$\displaystyle = 2\lim_{\theta\to 0} \frac{1}{\frac{\sin\theta}{\theta}} \cdot \frac{1+2(1)}{1}$

$= 2 \cdot 1 \cdot 3 = 6$

so $\displaystyle\lim_{\theta\to 0} \frac{2\theta + 4\theta\cos\theta}{\sin\theta\cos\theta} = 6$

(c) $\displaystyle\lim_{t\to\infty} \frac{\cos 2t}{t} = ?$

Solution

$-1 \leq \cos 2t \leq 1$ for any t

when $t \to \infty$, $\frac{1}{t} > 0$

Multiply each side by $\frac{1}{t} > 0$

$\displaystyle -\frac{1}{t} \leq \frac{\cos 2t}{t} \leq \frac{1}{t}$

$\displaystyle \lim_{t\to\infty} \frac{1}{t} = \lim_{t\to\infty} \frac{1}{t} = 0$

By Squeeze Theorem $\Rightarrow$

$\displaystyle \lim_{t\to\infty} \frac{\cos 2t}{t} = 0$

(d) $\displaystyle\lim_{x\to 1} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) =$

$= \sqrt{1^2+1} - \sqrt{1^2-1} = \sqrt{2} - \sqrt{0}$

$= \sqrt{2}$

$\displaystyle\lim_{x\to 1} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) = \sqrt{2}$

(e) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) =$

$\infty - \infty$

$\displaystyle = \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)\left(\sqrt{x^2+x} + \sqrt{x^2-x}\right)}{\sqrt{x^2+x} + \sqrt{x^2-x}}$

$\displaystyle = \lim_{x\to\infty} \frac{(x^2+x) - (x^2-x)}{\sqrt{x^2+x} + \sqrt{x^2-x}}$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2(1+\frac{1}{x})} + \sqrt{x^2(1-\frac{1}{x})}}$$

(note. $\sqrt{x^2} = |x| = x$
because when $x \to \infty$, $x > 0$)

$$= \lim_{x\to\infty} \frac{2x}{x\left(\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}\right)}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1$$

so
$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) = 1$$

---

(11) (a) $x^3+y^3 = 5xy \quad \Big/ \frac{d}{dx}$

$3x^2 + 3y^2y' = 5y + 5xy'$

$3y^2y' - 5xy' = 5y - 3x^2$

$y'(3y^2-5x) = 5y - 3x^2$

$$y' = \frac{5y-3x^2}{3y^2-5x}$$

(b) The tangent to the curve is horizontal iff its slope is zero.

$y' = 0$ iff $5y - 3x^2 = 0$

$$y = \frac{3x^2}{5}$$

$$x^3 + \left(\frac{3x^2}{5}\right)^3 = 5x \cdot \frac{3x^2}{5}$$

$$x^3 + \frac{27x^6}{125} = 3x^3$$

$27x^6 = 250x^3$

$27x^6 - 250x^3 = 0$

$x^3(27x^3 - 250) = 0$

$x = 0$ OR $x^3 = \frac{250}{27}$

$$x = \sqrt[3]{\frac{250}{27}} = \frac{5\sqrt[3]{2}}{3}$$

when $x=0$, $y=0$

when $x = \frac{5\sqrt[3]{2}}{3}$, $y = \frac{3}{5}\left(\frac{5\sqrt[3]{2}}{3}\right)^2$

$$y = \frac{5\sqrt[3]{4}}{3}$$

So the tangent is horizontal at $(0,0)$ and $\left(\frac{5\sqrt[3]{2}}{3}, \frac{5\sqrt[3]{4}}{3}\right)$

---

(12) (a) $g(x) = 2\left(\frac{2}{\sqrt{x}} + 1\right)^{-\frac{1}{3}}$

(NOTE: $(u^{-\frac{1}{3}})' = \frac{-1}{3}u^{-\frac{1}{3}-1} \cdot u'$
with $u = \frac{2}{\sqrt{x}} + 1$)

$$g'(x) = 2 \cdot \frac{-1}{3}\left(\frac{2}{\sqrt{x}}+1\right)^{-4/3}\left(\frac{2}{\sqrt{x}}+1\right)'$$

(NOTE. $\frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$)

$$g'(x) = \frac{-2}{3}\left(\frac{2}{\sqrt{x}}+1\right)^{-4/3}\left(2 \cdot \frac{-1}{2}x^{-\frac{1}{2}-1}\right)$$

$$g'(x) = \frac{2}{3}\left(\frac{2}{\sqrt{x}}+1\right)^{-\frac{4}{3}} x^{-3/2}$$

$$g'(x) = \frac{2}{3}x^{-\frac{3}{2}}\left(\frac{2}{\sqrt{x}}+1\right)^{-\frac{4}{3}}$$

(b) $y = \ln \dfrac{1}{t\sqrt{t+1}}$

$y = \ln 1 - \ln(t\sqrt{t+1})$

$y = \ln 1 - \ln t - \ln\sqrt{t+1}$

$y = 0 - \ln t - \frac{1}{2}\ln(t+1)$

$y' = -\frac{1}{t} - \frac{1}{2}\cdot\frac{1}{t+1}\,(t+1)'$

$y' = \frac{-1}{t} - \frac{1}{2(t+1)}$

$y' = \dfrac{-2(t+1)-t}{2t(t+1)} = \dfrac{-2t-2-t}{2t(t+1)}$

$y' = \dfrac{-3t-2}{2t(t+1)}$

(c) $h(x) = \cot(x^2)$

$\left((\cot u)' = \frac{-1}{\sin^2 u}\cdot u', \text{ with } u = x^2 \right)$

$h'(x) = \dfrac{-1}{\sin^2(x^2)}\,(x^2)'$

$h'(x) = \dfrac{-2x}{\sin^2(x^2)} = -2x\sin^{-2}(x^2)$

$h''(x) = (-2x)'\sin^{-2}(x^2) + (-2x)(\sin^{-2}(x^2))'$

$\left((u^{-2})' = -2u^{-3}\cdot u' \text{ with } u = \sin(x^2) \right)$

$h''(x) = -2\sin^{-2}(x^2) - 2x(-2)\sin^{-3}(x^2)\,(\sin(x^2))'$

$\left((\sin u)' = \cos u\cdot u' \text{ with } u = x^2 \right)$

$h''(x) = -2\sin^{-2}(x^2) + 4x\sin^{-3}(x^2)\cdot\cos(x^2)\cdot(x^2)'$

$h''(x) = -2\sin^{-2}(x^2) + 4x\sin^{-3}(x^2)\cos(x^2)\cdot 2x$

$h''(x) = \dfrac{-2}{\sin^2(x^2)} + \dfrac{8x^2\cos(x^2)}{\sin^3(x^2)}$

(d) $f(x) = \sin^{-1}(\cos(2x+5))$

$\left((\sin^{-1}u)' = \frac{1}{\sqrt{1-u^2}}\cdot u' \text{ with } u = \cos(2x+5) \right)$

$f'(x) = \dfrac{1}{\sqrt{1-\cos^2(2x+5)}}\cdot(\cos(2x+5))'$

$\left((\cos u)' = -\sin u\cdot u' \text{ with } u = 2x+5 \right)$

$f'(x) = \dfrac{-\sin(2x+5)\cdot 2}{\sqrt{\sin^2(2x+5)}}$

$f'(x) = \dfrac{-2\sin(2x+5)}{|\sin(2x+5)|}$

$f'(x) = \begin{cases} -2 & \text{if } \sin(2x+5) > 0 \\ 2 & \text{if } \sin(2x+5) < 0 \end{cases}$

(e) $y = x^{x+1}$ $\Big/ \ln$

$\ln y = \ln(x^{x+1})$

$\ln y = (x+1)\ln x$ $\Big/ \frac{d}{dx}$

$\frac{1}{y}\cdot y' = (x+1)'\ln x + (x+1)(\ln x)'$

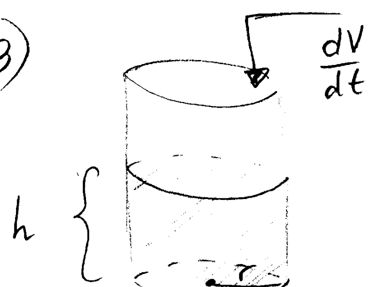
$\frac{1}{y}\cdot y' = \ln x + \frac{x+1}{x}$ $\Big/ \cdot y$

$y' = y\left(\ln x + \frac{x+1}{x}\right)$

$y' = x^{x+1}\left(\ln x + \frac{x+1}{x}\right)$

---

(13)



(a) $V'(t) = \frac{dV}{dt}$ is the rate of change of volume of water with respect to time.

$h'(t) = \frac{dh}{dt}$ is the rate of change of the height with respect to time.

Since the volume and height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero.

(c) $r = 10\,m$ $\qquad \frac{dh}{dt} = ?$

$\frac{dV}{dt} = 250$ L/min

Solution

$V = \pi r^2 h$ $\quad (m^3)$

$1\,m^3 = 1000\,L$ of water

$V = \pi(10)^2 \cdot 1000h$ $\quad (L)$

$\frac{dV}{dt} = 10^5\pi \frac{dh}{dt}$

$250 = 10^5\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{25}{10^4\pi}$ m/min

The water will rise at a rate of $\frac{25}{10^4\pi}$ m/min.

(14) $s(t) = 2t^3 - 5t^2 + t$

s = position (m)

t = time (s)

(a) let $v(t)$ = velocity

$v(t) = s'(t)$

$v(t) = 6t^2 - 10t + 1$

$v(2) = 6(4) - 10(2) + 1$

$v(2) = 5$ m/s velocity at $t = 2$ sec

(b) particle at rest when

$v(t) = 0$

$6t^2 - 10t + 1 = 0$

$t = \frac{10 \pm \sqrt{76}}{12}$ $\begin{cases} t \approx 0.64 \text{ sec} \\ t \approx 9.36 \text{ sec} \end{cases}$

The particle is at rest at about 0.64 sec. and 9.36 sec.

(c) let $a(t)$ = acceleration

$a(t) = v'(t) = 12t - 10$

$a(2) = 12(2) - 10$

$a(2) = 14$ m/s2 acceleration at $t = 2$ sec.

EXTRA CREDIT

(1) $V = \frac{P}{4n\ell}(R^2 - r^2)$

$V = \frac{PR^2}{4n\ell} - \frac{P}{4n\ell} r^2$

P, R, n, ℓ = constants

$\frac{dV}{dr} = \frac{-P}{4n\ell} \cdot 2r$

$\frac{dV}{dr} = \frac{-Pr}{2n\ell}$

(2) one point in common iff common tangent line

$f(x) = \ln x$ $\quad g(x) = cx^2$

$f'(x) = \frac{1}{x}$ $\quad g'(x) = 2cx$

let $x = a$ the point where they have a common pt

Then, $f'(a) = g'(a)$

$\frac{1}{a} = 2ca$

So, we need $\begin{cases} f(a) = g(a) \\ f'(a) = g'(a) \end{cases}$

$\begin{cases} \ln a = ca^2 \\ \frac{1}{a} = 2ca \end{cases} \Rightarrow ca^2 = \frac{1}{2}$

$\ln a = \frac{1}{2} \Rightarrow a = e^{\frac{1}{2}}$

if $a = e^{\frac{1}{2}}$, $c(e^{\frac{1}{2}})^2 = \frac{1}{2}$

$ce = \frac{1}{2}$, $c = \frac{1}{2e}$

(3) $$\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x} = \lim_{x\to 0}\frac{(\sqrt{ax+b}-2)(\sqrt{ax+b}+2)}{x(\sqrt{ax+b}+2)}$$

$$= \lim_{x\to 0}\frac{(ax+b)-4}{x(\sqrt{ax+b}+2)} = 1$$

Note that the denominator approaches 0 as $x\to 0$
Therefore, the limit will exist only if the numerator also approaches 0 as $x\to 0$.

We need $a(0)+b-4=0 \Rightarrow b=4$

The equation becomes

$$\lim_{x\to 0}\frac{ax+4-4}{x(\sqrt{ax+4}+2)} = 1$$

$$\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} = 1$$

$$\frac{a}{4} = 1 \Rightarrow a = 4$$

So, $a=b=4$